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FINAL REPORT

A STUDY OF THE STABILITY OF REINFORCED CYLINDRICAL AND CONICAL SHELLS SUBJECTED TO VARIOUS TYPES AND COMBINATIONS OF LOADS

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Submitted to

George C. Marshall Space Flight Center

National Aeronautics and Space Administration

NASA Contract NAS 8-5168

October 15, 1962 - January 14, 1964

Project Directors: Carl C. Steyer and William K. Rey

UNIVERSITY OF ALABAMA BUREAU OF ENGINEERING RESEARCH

and

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Introduction

Preliminary theoretical and experimental studies of the strength and stability of cylindrical and conical shells were performed by the University of Alabama under the terms of Contract Number DA-01-009-ORD-334 with the Redstone Arsenal and Contract Number DA-01-009-ORD-866 with the U.S. Army Ordnance District, Birmingham, Alabama. As a result of these preliminary studies and discussions with personnel of the Strength Analysis Branch of the Propulsion and Vehicle Engineering Division at the George C. Marshall Space Flight Center, a long range research program was formulated for the purpose of providing analytical procedures, design data and digital computer programs for the analysis and design of cylindrical and conical shells that could be included in a space vehicle structures handbook. The first research effort designed to achieve this purpose was accomplished under the terms of contract NAS 8-5012 between the George C. Marshall Space Flight Center and the University of Alabama during the period from May 28, 1962 to October 15, 1962. The results of this initial effort under the terms of contract NAS 8-5012 were submitted to the GCMSFC as a University of Alabama Bureau of Engineering Research Summary Report in four sections as follows: Section 1 - "General Instability of An Orthotropic Circular Cylindrical Shell

Subjected to A Pressure Combined with An Axial Load Considering Both Clamped and Simply Supported Edge Conditions" by Carl C. Steyer and Thomas A. Carlton, Jr.; Section 2 - "Stress in A Segment of A Conical Shell Subjected to Lateral Normal Load" by Chin Hao Chang; Section 3 - "General Instability of An Orthotropic Circular Conical Shell Subjected to Hydrostatic Pressure and A Compressive Axial Force" by Carl C. Steyer and Shih-Cheng Zien; and Section 4 - "Matrix Shear Lag Analysis Utilizing a High-Speed Digital Computer" by William K. Rey. Abstracts of these four reports appeared in Volume 2, Number 2 issue of the Scientific and Technical Aerospace Reports as abstract numbers N64-11335, N64-11336, N64-11337 and N64-11332 respectively.

The initial effort of contract NAS 8-5012 was continued and expanded under the terms of contract NAS 8-5168 which provided for a twelve month effort beginning October 15, 1962. Modification Number 2 extended the period of performance through December 14, 1962 and Modification Number 3 extended the period of performance through January 14, 1964.

A letter of appointment dated November 5, 1962 from Marion S. Hardee, Contracting Officer, designated Mr. Norman C. Schlemmer and Mr. James B. Sterett of the Propulsion and Vehicle Engineering Division, Structures Branch, as his principal and alternate representatives, respectively. Amendment Number 1 executed by James W. Fletcher, Contracting Officer, and dated August 30, 1963 relieved Mr. James B. Sterett of this responsibility and appointed Mr. Orville E. Wheeler and Mr. Norman C. Schlemmer as the principal and alternate representatives, respectively, of the Contracting Officer.

Scope of Work

The work scope of contract NAS 8-5168 provided for a study of the following seven items:

1. Completion of the theoretical studies in the evaluation and application of Bodner's work to stiffened cylinders subjected to a pressure, an axial load, or appropriate combinations of these loads with an experimental verification of the results of these studies.

2. A theoretical and experimental study of a very thin orthotropic cylinder that buckles in a diamond shaped pattern as a result of being subjected to an axial load, a bending moment, a pressure, a torque, or certain combinations of these loads.

3. The development and experimental verification of a linear differential equation expressing the instability of a cylinder of a type similar to the one developed by Bodner but which includes additional non-linear or second order terms in the strain-displacement relationships.

4. A theoretical and experimental study of a stiffened cylinder or cone frustum subjected to a bending moment or a combination of a bending moment and other loads.

5. The analysis of mathematical problems presented by the strength and instability studies.

6. The theoretical analysis of a segment of a cone frustum considering temperature distribution and other loads as specified by the Government.

7. The experimental and theoretical study of the stress distribution and shear lag for stiffened cylinders, cones or cone frustums.

Personnel

Professional personnel of the University of Alabama participating in the accomplishment of the work scope were Dr. T. A. Carlton, Jr., Dr. C. H. Chang and Dr. C. C. Steyer of the Department of Engineering Mechanics, Professor William K. Rey of the Department of Aerospace Engineering and Mr. William S. Viall of the Research Institute. The following students in the College of Engineering served as Graduate Associates, Graduate Assistants or Student Technicians: Thomas D. Easter, Thomas C. Evans, Raymond C. Montgomery, Colonel M. Pearson, Charles H. Ratcliff, Melvin K. Richardson, Jimmie L. Smith, Charles R. Weeks and Tao Wu. Secretarial assistants and machinists of the Bureau of Engineering Research, the Research Institute, the Department of Aerospace Engineering and the Department of Engineering Mechanics were utilized.

Summary of Results

I. The analytical study of the instability of circular cylindrical shells was continued. Equation 32 in Section 1 of the Summary Report for contract NAS 8-5012 was programmed and checked. The Fortran II computer program for the solution of this equation was previously submitted as Technical Report A for contract NAS 8-5168 and is included in this report as Appendix A. This program may be used to predict the instability of a short orthotropic or stiffened circular cylindrical shell subjected to a combination of external pressure and axial loads with either clamped or simply supported edge conditions.

II. A second study of the stability orthotropic circular cylinders considered the case of a cylinder simultaneously subjected to an axial load, an end moment and a uniform radial pressure. This study was supported jointly by NASA contract NAS 8-5168 and NASA research grant NsG-381. The results of this study were previously submitted as Report Number 11 of the University of Alabama Research Institute and are included in this report as Appendix E. Appendix E contains both the Analysis and the Fortran II computer program.

III. The analytical study of the stability of orthotropic circular conical shells was continued with the programming in Fortran II language of the constants and coefficients that were presented in Section 3 of the Summary Report for contract NAS 8-5012. This program was previously submitted as Technical Report B for contract NAS 8-5168 and is included in this report as Appendix B.

IV. A second study of the stability of conical shells considered the case of a segment of an isotropic truncated conical shell with linearly varying thickness subjected to lateral normal loads. An asymptotic general solution was obtained and previously submitted as Technical Report C for contract NAS 8-5168. The results of this analysis are contained within this report as Appendix C.

V. A survey of current literature was conducted to identify publications containing information pertaining to the subject matter of contract NAS 8-5168. The Scientific and Technical Aerospace Reports issued by the Scientific and Technical Information Division of NASA, the Technical Abstract Bulletin issued by the Defense Documentation Center, the Applied Mechanics Reviews published by the

American Institute of Aeronautics and Astronautics were scanned each month for reports that appeared to contain information of value. Copies were then obtained of those articles whose title indicated they were related to work scope of contract NAS 8-5168. Abstracts were then prepared of those publications that contained particularly useful data or information. During the period of performance, copies were obtained of 283 articles and abstracts were prepared of 39 articles. The list of publications and abstracts were previously submitted as Technical Report D for contract NAS 8-5168 and are contained within this report as Appendix D.

VI. The possibility of experimentally verifying some of the analytical studies of the stability of cylindrical shells was considered. After a comprehensive review of available literature pertaining to the fabrication and testing of plastic models, it became apparent that no plastic material or fabrication procedure had been widely adopted for model testing. Every procedure appears to contain inherent faults for the study being contemplated. Essentially the problem becomes that of selecting the procedure whose known faults may be expected to have the least effect on the tests to be conducted. Although no cylinders were actually made or tested, cellulose acetate and vinyl were selected as the two materials that would be used in preliminary studies. These two materials are available as sheet in various sizes and thicknesses. This experimental program will be continued under the terms of contract NAS 8-11155.

VII. In Section 4 of the Summary Report for contract NAS 8-5012, two analyses were presented for determining the stress distribution in an axially loaded, integrally stiffened panel. A number of other analyses of this problem are available. In general, the differences in the results obtained from the various analyses may be attributed to the differences in the assumptions employed. In order to determine the validity of the various analyses, an experimental program was begun. This experimental program employs integrally stiffened 7075-T651 aluminum alloy panels approximately eighteen inches wide and twenty-four inches long. Four panels were prepared having a ratio of stiffener area to stringer area varying from one-half to two.

Preliminary testing of the first panel, Panel A, indicated that poor machining had produced a panel that could not be effectively utilized. However, the preliminary testing of Panel A did disclose many problems in the instrumentation and certain undesirable characteristics of the testing machine used to apply loads to the panels. A considerable amount of time and effort was necessarily expended in refining the instrumentation, correcting some of the deficiencies in the testing machine and obtaining satisfactory end supports.

Preliminary testing of the second panel, Panel B, indicated that all of the known problems had been corrected. Panel B was instrumented with a total of one hundred and forty strain gauge channels. Two complete sets of data were obtained for each of four different symmetrical loading conditions. Computer programs were written for reducing the strain gauge data obtained from both the uniaxial and rosette gauges utilizing the UNIVAC SS 80 computer.

After reducing the data obtained from testing Panel B it became apparent that the results were of no value for a number of reasons. The results obtained from two supposedly identical tests for each of the loading conditions did not agree in many important instances. Secondly, although every precaution was taken to eliminate bending of the panel, bending was present in some of the tests and an insufficient number of strain gauge stations prevented a full evaluation of the bending effect. Finally, some of the results exhibited an unexplained non-linearity and apparent zero drift in certain gauge channels. Therefore, the data obtained in the testing of Panel B is not being submitted at this time pending a complete re-evaluation of the testing procedure, instrumentation, and data reduction methods.

The attempt to obtain satisfactory, reproducible experimental data for Panel B, as well as for Panels C and D, will be continued under the terms of contract NAS 8-11155. The experimental results will be compared with various analytical predictions in an attempt to determine the most satisfactory analysis.

APPENDIX A

FORTRAN II COMPUTER PROGRAM FOR THE EVALUATION OF
A DONNELL TYPE OF DIFFERENTIAL EQUATION FOR A
SIMPLY-SUPPORTED CYLINDRICAL SHELL

Prepared by
Thomas D. Easter

This appendix was previously submitted as Technical Report A for contract NAS 8-5168.

Technical Report A for NASA Contract NAS8-5168

FORTRAN II COMPUTER PROGRAM FOR THE EVALUATION OF A DONNELL TYPE OF DIFFERENTIAL EQUATION FOR A SIMPLY-SUPPORTED CYLINDRICAL SHELL

24841

Prepared By

Thomas D. Easter

This report presents a FORTRAN II Computer Program for use with the Univac Solid-State 80 Computer for obtaining the solution of equation (32) in the report, "A Study of the Stability of Reinforced Cylindrical and Conical Shells Subjected to Various Types and Combinations of Loads, Section I-General Instability of an Orthotropic Circular Cylinderical Shell Subjected to a Pressure Combined with an Axial Load Considering Both Clamped and Simply Supported Edge Conditions," by Carl C. Steyer and T. A. Carlton, Jr., submitted November 1962 under Contract No. NAS8-5012 to the George C. Marshall Space Flight Center of the National Aeronautics and Space Administration.

The FORTRAN II program solves equation (32) of the above referenced report by the following computational steps:

<u>Part A:</u> <u>For Assigned</u> $^{R}/t$ and λ

1. Given particular values of E_x, E_s, ν_{xs}, ν_{sx}, and h (or t), the extensional and shearing stiffnesses, α_1, α_2, α_3, α_4, and the bending and twisting rigidities, D_1, D_2, D_3, D_4, are calculated.
2. Using the computed values of Step 1, the constants d_1, through d_{42}, are computed.
3. For a selected set of values for m and n, the constants d_{43} through d_{49} are computed.

4. Using an assigned value of k_1 ($k_1 = \frac{p}{q}$), the solutions of equation (32) yields maximum and minimum values of R^2q. Using the minimum positive value of q, the value of σ is computed.

5. The value of m is incremented and Steps 3 and 4 are repeated yielding a new value of σ.

6. Step 5 is repeated until $\sigma = \sigma_{min}$ is obtained for the n assumed in Step 3 and k_1 of Step 4. The value of m is then set to its original value.

7. The value of n is incremented and Steps 3 through 6 are repeated.

8. Step 7 is repeated each time n is incremented until $\sigma = \sigma_{min}$, the minimum σ occurring for the value of k_1 assumed in Step 4 is obtained. The value of n is then returned to its original value.

9. The value of k_1 is incremented and Steps 3 through 8 are repeated resulting in a value of $\sigma = \sigma_{min}$ for the new value of k_1.

10. Step 9 is repeated for an applicable range of k_1 values, rendering a $\sigma = \sigma_{min}$ for some combinations of m, n and k_1 and the assumed values of R/t and λ. It should be noted that k_1 may be positive or negative. At the completion of Step 10 the value of k_1 is returned to its original value.

Part B: For a New R/t Value

1. Since a new value of R/t for a fixed value of R implies a change in t, new values of α_1, α_2, α_3, α_4, D_1, D_2, D_3, and D_4, are computed.

2. Steps 2 through 10 of Part A are repeated resulting in a minimum positive value of σ for some combination of m, n and k and the new R/t.

The information obtained from Parts A and B may be presented in tabular form involving σ, m, n, k_1, R, t, and L to display composite results of the effects of cylinder geometries and internal pressures or the information may be displayed in graphical form, e.q., σ vs $^R/t$, etc.

A summary of the FORTRAN names used in the computer program is shown below in the Fortran Notation Legend.

FORTRAN NOTATION LEGEND

Variable	Fortran Program Name
d_1	D(1)
d_2	D(2)
d_3	D(3)
d_4	D(4)
.	.
.	.
.	.
d_{49}	D(49)
α_1	A 1
α_2	A 2
α_3	A 3
α_4	A 4
D_1	D 1
D_2	D 2
D_3	D 3
D_4	D 4
λ	Y

Variable	Fortran Program Name
m	A
n	B
k_1	S
E_x, E_s, (Moduli of Elasticity for orthotropic circular cylindrical shell)	
Modulus of ridigity	G
ν_{xs}, ν_{sx}, (Poisson's ratios for orthotropic shell)	FNUXS, FNUSX
K_1 ($^P/q$)	S 1
R (radius)	R
L (length)	C
h or t (thickness)	H
$-\frac{d_{49}}{d_{46}}$	RSQ
$\pi -\frac{d_{49}}{d_{46}}$	PRSQ
$\frac{PRSQ}{2\pi RH}$	SGMA
integer counters used for program purposes	I & J
$^R/_t$ Value	ROT
$(d_{46} + k_1 \; d_{47})$	BR
$d_{46} + k_1 \; d_{47}$	BRABS
$d_{49} k_1 d_{43}$	AC

$(BR)^2-4(AC)$	ROT
RAD	RADRT
$(R^2q)_1$	RSQ1
$(R^2q)_2$	RSQ2
$\pi(R^2q)_1$	PRSQ1
$\pi(R^2q)_2$	PRSQ2
$\frac{PRSQ1}{2\pi RH}$	SGMA1
$\frac{PRSQ2}{2\pi RH}$	SGMA2
R^2	R2
$\alpha^2{}_4$	A42
$\alpha_1\alpha_2$	A12
$\alpha_1\alpha_3$	A13
$\alpha_2\alpha_3$	A23
$\alpha_3\alpha_4$	A34
$\alpha_1 D_3$	A1D3
$\alpha_2 D_3$	A2D3
$\alpha_3 D_2$	A3D2
$D_1 R$	D1R
$D_2 R$	D2R
$D_3 R$	D3R
$D_4 R$	D4R
k^2	S2
$\lambda^2 m^2$	YA2

$\lambda^4 m^4$	YA4
n^2	B2
n^4	B4
$D_3 D_4$	D34
$D_2 D_3$	D23

The following explanations of the program input and output system used the Fortran Program names listed above in order to make the explanations compatible with the actual Fortran Program.

INPUT

Data cards are read into the program in the following order and form: The First Data Card contains the constants EX, ES, FNUXS, G, and the variable H. They are in the following order, EX, ES, G, H, FNUSX, and FNUXS, with a Read Format Statement of (6E10.4), which means this data must be punched on the first data card in the following form.

Columns 01 to 10 First data word, EX.
Columns 11 to 20 Second data word, ES.
Columns 21 to 30 Third data word, G.
Columns 31 to 40 Fourth data word, H.
Columns 41 to 50 Fifth data word, FNUSX.
Columns 51 to 60 Sixth data word, FNUXS.

Example: Let EX = 1.03×10^7, and ES = 1.3×10^2. Only the first portion of the data card is illustrated below. The remaining values, G, H, FNUSX, FNUXS, would be punched in the remainding spaces as outlined above with the same form as EX and ES.

```
                      EX                        ES
Column No. 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20--80
Word Form    1 . 0 3 + 0 7        -  1  .  3  -  0  2
            Exponent sign    Algebraic sign     Exponent
```

G, H, FNUSX, and FNUXS would go in columns 21 to 60, following EX and ES.

The Second Data Card contains the variables S, H, R, C, S1, and ROT. S1 here is the initial value of S1. The Read Format Statement is (5E10.4, I4). I4 is the only change from the above explanation. It means begin at column no. 51, and punch an integer of 1 to 4 numbers:

Example: For ROT = 25 and ROT = 1250.

```
                                    ROT
Column No -------------------51 52 53 54 55-------------------
          -------------------    2  5      -------------------
          ------------------- 1  2  5  0   -------------------
```

The Third Data Card contains the variables S1. S1 here is the second value of S1. The Read Format Statement is (E15.7) which means one S1 value per card punched in the first 15 spaces.

The number of different values of S1's desired for each R/t ratio will determine the number of remaining data cards which will have the same form as the third data card explained above.

Example: If S1 = 0.1, 0.2, 0.3, 0.4, 0.5 for an assigned value of the R/t ratio, the initial value of S1 = 0.1 will go on the second data card, the remaining values of S1 will go on data cards 3 thru 6 as follows: 0.2 to card three, 0.3 to card four, 0.4 to card five and 0.5 to card six. The form used is the same as that explained for card three above.

If it is desired to run data for more than one R/t ratio, duplicate the above data card procedure for the new R/t ratio and place these cards after the cards for the first R/t ratio. This can be carried out for any number of R/t ratios.

OUTPUT

In the printout, the symbol # represents an equal sign (=). The variable is printed out, followed by its value. The meaning of the variable can be found in the Fortran Notation Legend.

Exponent sign ↴ Exponent
Example: A # 1.0000E 00 In the Fortran Notation Legend, A is shown to represent m, therefore the above statement reads m = 1. If the exponent is positive, a blank space will be left between the E and the exponent as shown above. If the exponent is negative, a minus sign (-) will be printed between the E and the exponent. If m had been negative, a minus sign would be printed between the # and 1 as follows: A # - 1.0000E 00

When "SCALED RADICAL OPERATION" appears on the printout, this means that some of the values in the operation have reached the point of overflow and have been scaled down.

When "NEGATIVE RADICAL" appears on the printout, this means that the value under the radical in the quadratic equation used in the program, is negative. Conversely, the printout "POSITIVE RADICAL" indicates that the value is positive.

"SIGMA MINIMUM" is the minimum value for σ for a particular value of m, with m varying and n held constant. "SIGMA MINIMUM FINAL" is the final minimum value of σ for a k_1 value, varying m over a range of n's with k_1 and R/t held constant. The final value printed for "SIGMA MINIMUM FINAL" is the minimum value.

Each time new values of A, B, ROT, and S1 are used, it will be indicated on the printout as "NEW VALUE FOR A" or "NEW VALUE FOR B," etc.

When S1 = 0, this is indicated on the printout and the program performs a special operation for finding the value of σ in this case, which will be indicated by "SGMA". Only one value of σ is found, since the quadratic equation is not used in this special case.

CONTROL OF RANGE OF A AND B

Statement 134 in the program controls the range over which B will run. This statement is in the following form:

```
Column No. 1 2 3 ----7 8 9 10 ---------------------------80
           1 3 4      I F    ( B - 17.0) 127, 130, 130
```

The number 17.0 inside the parenthesis controls the upper limit of B. In this case B will run or increment up to 17. If it is desired to change the upper limit of B, all that is needed is to change the number in parenthesis, which must be written as a real number.

Example: For the upper limit to be 24, change 17.0 to 24.0.

```
Column No. 1 2 3 ----7 8 9 10 ---------------------------80
           1 3 4      I F    ( B -24.0) 127, 130, 130
```

The lower limit of B can be changed by changing statement 100, which is in the following form:

```
Column No. 1 2 3 ----7 8 9 10 11 12 13 14 --------------80
           1 0 0      B = 1  .  0
```

The lower limit of B, or the point B starts at is 1. To have B start at 4, you would change the 1.0 to 4.0. These must be written as real numbers. The new statement would be as follows:

```
Column No. 1 2 3 ----7 8 9 10 11 12 --------------------80
           1 0 0      B = 4  .  0
```

In the above examples, B would start at 4 and run to 24. In the original case, B would start at 1 and run to 17.

The upper limits of A is controlled by statement 135, which is in the following form:

```
Column No. 1 2 3 ----7 8 9 10 11 -----------------------80
           1 3 5      I F    ( I - 25) 102, 126, 126
```

The number 25 inside the parenthesis means that A will increment by 1, 25 times after a minimum value of σ has been found. The total number of A's will be 25 plus the number of A's used to reach a minimum value of σ. To change the upper limit of A, change the number in the parenthesis. These numbers must be written as integers only. If the value 25 were replaced by 30, 5 more values of A would be added. The new statement would be as follows:

```
Column No. 1 2 3 -------7 8 9 -------------------- -----80
           1 3 5        I F  (I - 30) 102, 126, 126
```

The lower limit of A is controlled by statement 101, which is of the following form:

```
Column No. 1 2 3 ----7 8 9 10 11 -----------------------80
           1 0 1     A = 0 .  0
```

This means that A starts at 0 and must be written as as a real number 0.0. If you wanted to start A at 6, you would change the statement to read:

```
Column No. 1 2 3 ----7 8 9 10 11 -----------------------80
           1 0 1     A = 6 .  0
```

By changing statements 134, 135, 100, and 101, the programmer can change or cut down the range of A and B, thus eliminating needless computations in certain cases.

The source program listing for the cylinder problem is as follows:

```
C     FORTRAN PROGRAM FOR CYLINDER PROBLEM.  A FINAL MINIMUM VALUE OF
C     SIGMA IS FOUND FOR A FAMILY OF K1 VALUES FOR EACH RADIUS TO
C     THICKNESS RATIO.
      DIMENSION D(49)
99    READ 1, EX. ES, G, H, FNUSX, FNUXS
1     FORMAT (6E10.4)
      A1=(EX*H)/(1.0-FNUSX*FNUXS)
      A2=(ES*H)/1.0-FNUSX*FNUXS)
      A3=(G*H)/2.0
      A4=(EX*FNUSX*)/(1.0-FNUSX*FNUXS)
      D1=(EX*H**3)/(12.0*)1.0-FNUSX*FNUXS))
      D2=(ES*H**3)/(12.0*(1.0-FNUSX*FNUXS))
      D3=(G*H**3)/12.0
      D4=(EX*FNUSX*H**3)/12.0*(1.0 FNUSX*FNUSX))
      READ 2, S, H, R, C, S1, IROT
2     FORMAT (5E10.4,I4)
      PRINT 17, IROT
17    FORMAT (13/. 37H STARTING OVER FOR A NEW VALUE OF ROT = ,I4:
      PRINT 3, A1, A2, A3, A4, D1, D2, D3, D4, S, H, R, C, S1, IROT
3     FORMAT (18H DATA PRINT OUT- - - -,2 5HA1 = ,E15.7,/,5HA2 = ,E15.7,/,
     15HA3 = ,E15.7,/,5HA4 = ,E15.7,/,5HD1 = ,,E15.7,/,5HD2 = ,E15.7,/,
     25HD3 = ,E15.7,/,5HD4 = ,E15.7,/,5HS = ,E15.,/,5HH = ,E15.7,/,5H
     3R = , E15.7,/,5HC = ,E15.7/,5HS1 = .E15.7,/,6HROT = ,14.5/)
      Y=(3.1415926*R)/C
      R2=(R**2)
      A42=(A4**2)
```

```
D34=D3*D4
D23=D2*d3
A13=A1*A3
A12=A1*A2
A23= A2*A3
A34=A3*A4
A1D3=A1*D3
A2D3=A2*D3
A3D2=A3*D2
D1R=D1*R
D2R=D2*R
D3R=D4*R
D4R=D4*R
I=0
SMAX2=1.0E+49
D(41)=A12*R2
D(42)=(-A42*R2)+A1*D2
D(39)=(A23*R2)+(A2D3/2.0)
D(40)=(A3D2)+(D23/(2.0*R2))
D(34)=(A12*A3*R*R2)-(A43*A3*R*R2)-((A12+A42)*D3R/2.0)
D(35)=(A13*D2R)+((A1*D23)/(2.0*R))
D(36)=2.0*A1*D4
D(37)=A1*R
D(38)=D(37)
D(33)=((A34+A42-A12)*2.0*R2)-A4*D3)
D(27)=2.0*A34*D3R
D(28)=((A12-A42-(2.0*A34))*D2R)+(((2.0*A3)+A4)*D23/R)
D(29)=((A12-A34-A42)*2.0*R2)-(A4*D3)
```

```
D(30)=(((A3*D4)+A1*D2))*2.0+(D34/R2)

D(32)=A3*R)+(D3/(2.0*R))

D(31)=-D(32)

D(25)=-(2.0*D(39))

D(26)=(((2.0*A3)-A1)*R)+(D3/R)

D(22)=(A23*D2R)+((A2*D23)/(2.0*R))

D(23)= D(25)

D(24)=2.0*D(40)

D(19)=(2.0*A13*D4R)+((A1*D34) R)

D(20)=A1*D1

D(21)=A13*R2+A1D3/2.0

D(18)=A1D3-(2.0*A13*R2)

D(14)=(A12-(2.0*A34)-A42)*2.0*D3R

D(15)=(((2.0*A12)-(4.0*A34)-2.0*A42))*D4R)+(2.0*A13*D2R+1(((A1*D23)+
(4.0*A3*D34)+(2.0*A4*D34))/R)

D(16)=(A13*R2+(A1*D4)+(A1D3/2.0)

D(17)=((A12-A42-(2.0*A34))*R2+(((2.0*A3)+A4)*D3)

D(13)=((2.0*A34)-A12+A42)*2.0**R2

D(9)=(4.0*A23*D3R)+((2.0*A4*D23)/$)

D(10)=((A12-A34-A42)*2.0*D2R)+(2.0*A23*D4R)+(((2.0*A3*D23)
1+(A2*D34))/R)

D(11)=((A12-A42-(2.0*A34))*R2)+(A1*D2)-A4*D3)+(2.0*A3*D4)+(D34/R21)

D(12)=D*39)

D(8)=D(25)

D(6)=2.0*D(22)

D(7)=(A23*R2)+A3D2+((A2D3+(D23/R2))/2.0)

D(5)=A13*D1R)+((A1D3*D1)/(2.0*R))
```

```
      D(4)=((A12-A42-(2.0*A34))*D1R)+(2.0*A13*(D3R+D4R))+(((A1*D34)+1(2.0*
      A3*D1*D3)+(A4*D1*D3))/R)
      D(3)=2.0*A12*D3R+2.0*A13*D4R+A23*D1R+A13*D2R-2.0*A42*D3R
     1-4.0*A34*D3R-2.0*A42*D4R-4.0*A34*D4R+(A2D3*D1+A1*D23)/(2.
     20*R)+(4.0*A3*D34+2.0*A4*D34)/R
      D(2)=(((2.0*A12)-A42-(2.0*A34))*D2R)+(2.0*A23*(D3R+D4R))
     1+(((A2*D34)+(2.0*A3*D23)+A4*D23))/R)
      D(1)=(A23*D2R)+((A2*D23)/(2.0*R))
      PRINT 4, (J, D(J), J = 1, 42)
4     FORMAT (2HD(,12,4H) = ,E15.7/)
100   B=1.0
101   A=0.0
      PRINT 5,
5     FORMAT (14HNEW VALUE OF B,2/)
      SMAX=1.0E+49
102   PRINT 18,
18    FORMAT (14 HNEW VALUE OF A,2/)
      A=A+1.0
      YA2=Y**2*A**2
      YA4=YA2**2
      B2=B**2
      B4=B2**2
      S2=S**2
      D(43)=((YA4*D(37))+(YA2*B2*D(32)))/2.0
      D(46)=-((((YA2*YA4*D(21)+YA4*B2*D(17)+YA2*B4*D(12)))/2.0)
      D(47)=(YA2*(D(41)+(S2**D(42))))-(B2*(D(39)+(S2*D(40))))+(YA4*S
     1*D(36))+(YA2*B2*(D(29)+(S*D(30))))+(B4*(D(23)+(S*D(24))))-(YA4
```

```
     2*YA2*D(D(20))-(YA4*B2*D(16))-(YA2*B4*D(11))-(B2*B4*D(7))
      D(49)=(YA4*(D(34)+(S2*D(35))))+(YA2*B2*(D(27)+(S2*D(28))))+
     1(B4*S2*D(22))-(YA2*YA4*S*D(19))YA4*B2*(D(14)+(S*D(15))))-YA2
     2*B4*(D(9)+(S*D(10))))-(B2*B4*S*D(6))+(YA4*YA4*D(5))+(YA2*YA4
     3*B2*D(4))+(YA4*B4*D(3))+(YA2*B4*B2*(2))+(B4*B4*D(1))
      PRINT 6, D(43), D(46), D(47), D(49)
6     FORMAT (8HD(43) = .E15.7,/,8HD(46) = ,E15.7,/,8HD(47) = ,E15.7,/,
     18HD(49) = ,E15.7,6/)
      IF (S1-0.0) 103, 104, 103
104   PRINT 7
      FORMAT (8HS1 = 0.0)
      RSQ=(-(D(49)/D(46)))
      PRSQ=3.1415926*RSQ
      SGMA=PRSQ/(6.2831852*R*H)
      PRINT 8, SGMA
8     FORMAT (HSGMA = ,E15.7,12H FOR S1 = 0.0)
      IF (SGMA) 150, 151, 151
150   PRINT 20,
20    FORMAT (30HSGMA FOR S1=0.0 IS NEGATIVE)
      GO TO 102
151   SGMAM=SGMA
      GO TO 122
103   BR=D(46)+S1*D(47)
      BRABS=ABS(BR)
      DOM=2.0*S1*D(43)
      IF (BRABS-1.0E+10) 105,105, 106
106   BR=BR*(1.0E-10)
```

```
      D(49)=D(49)*(1.0E-10)
      D(43)=D(43)*(1.0E-10)
      AC=D(49)*S1*D(43)
      D(49)=D(49)*(1.0E+10)
      D(43)=D(43)*(1.0E+10)
      RAD=BR**2-4.0*AC
      IF (RAD) 107, 108, 108
107   PRINT 9,
9     FORMAT (24HSCALED RADICAL OPERATION)
      GO TO 109
108   RADRT=SQRT(RAD)
      RADRT=RADRT*(1.0E+10)
      BR=BR*(1.0E+10)
      PRINT 10,
10    FORMAT (24HSCALED RADICAL OPERATION)
      GO TO 111
105   AC=D(49)*S1*D(43)
      RAD=BR**2-40*AC
      IF (RAD) 109, 110, 110,
109   PRINT 11, Y, RAD, S1, A, B,
11    FORMAT (26HNEGATIVE RADICAL----Y = ,E15.7,15X6H Rad = ,E15.7,15X
     15H S1 = ,E15.7, 15X4 HA = ,E15.7, 15X4 HB = ,E15.7,6.)
      GO TO 102
110   RADRT=SQET(RAD)
111   RSQ1=(( BR)+(RADRT))/DOM
      RSQ2=((-BR)-RADRT))/DOM
      PRSQ1=(3.1415926*RSQ1)
```

```
      PRSQ2=(3.1415926*RSQ2)
      SGMA1=PRSQ1/(6.2831852*R*H)
      SGMA2=PRSQ2/(6.2831852*R*H)
      PRINT 12, Y, RSQ1, RSQ2, PRSQ2, S1, A, B, SGMA1, SGMA2
12    FORMAT (26H POSITIVE RADICAL--Y = ,E15.7, 15X7HRSQ1 - ,E15.7, 15X
     17HRSQ2 = .E15.7, 15X8HPRSQ1 = ,E15.7, 15X8HPRSQ2 = ,E15.7, 15X5HS1 =
     2,E15.7, 15X4 HA = ,E15.7, 15X4HB = ,E15.7, 15X8H SGMA1 = ,E15.7,
     15X8HSG=MA2 = ,E15.7,6/)
      IF (SGMA1-SGMA2) 113, 112, 112
112   IF (SGMA2) 114, 115, 115
114   IF (SGMA1) 116, 119, 119
116   PRINT 13
13    FORMAT (33HBOTH SGMA1 AND SGMA2 ARE NEGATIVE)
      GO TO 102
115   SGMAM#SGMA2
      GO TO 122
113   IF (SGMA1) 118, 119, 119
118   IF (SGMA2) 120, 115, 115
120   PRINT 13,
      GO TO 102
119   SGMAM=SGMA1
122   IF (SGMAM=SMAX) 123, 124, 124
123   SMAX=SGMAM
      GO TO 102
124   U=A-1.0
      PRINT 14, SMAX, U, B, S1, IROT
14    FORMAT (15H SGMA MINIMUM = ,E15.7, 10H, FOR A = E16 7, 5X4HB =
```

```
    1E15.7, 5X5 HS1 = ,E15.7, 5X6HROT = ,14.6,/)
     I=I+I
135  IF (1-5) 102, 126, 126
126  I=0
134  IF (B-2) 127, 130, 130
127  IF (SMAX-SMAX2) 129, 128, 128
128  B=B+1.0
     GO TO 101
129  PRINT 15, SMAX, A, B, S1, IROT
15   FORMAT (22HSIGMA MINIMUM FINAL = ,E15.7, 10H, FOR A = ,E15.7,5X
    14HB = ,E15.7, 5X5HS1 = .E15.7, 5X6HROT = ,14.6/)
     B=B+1.0
     SMAX2=SMAX
     GO TO 101
130  READ 22, S1
22   FORMAT (E15.7)
     IF (S1-9.999999E+49) 131, 99, 131
131  PRINT 16
16   FORMAT (36HSTARTING OVER FOR A NEW VALUE OF S1.)
     GO TO 100
```

APPENDIX B

FORTRAN II COMPUTER PROGRAM FOR THE EVALUATION OF A DONNELL TYPE OF DIFFERENTIAL EQUATION FOR AN ORTHOTROPIC CIRCULAR CONICAL SHELL

Prepared by

Thomas D. Easter

Colonel M. Pearson

Melvin K. Richardson

This appendix was previously submitted at Technical Report B for contract NAS 8-5168.

Technical Report B for NASA Contract NAS8-5168

FORTRAN II PROGRAM FOR THE EVALUATION OF A DONNEL TYPE OF DIFFERENTIAL EQUATION FOR AN ORTHOTROPIC CIRCULAR CONICAL SHELL

24042

Prepared by

Thomas D. Easter, Research Assistant

and

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This report presents a FORTRAN II Computer Program that operationally is compatible with the FORTRAN processors of the IBM 7090 and the UNIVAC SS 80 computers. Additional details regarding the individual processors, actual machine compilation and object program execution, and so forth, are available in separate programming and operations reference manuals. Specifically, this report presents a FORTRAN II Computer Program that evaluates constants and coefficients for the Donnel-type, eighth-order, differential equation in the report, "A Study of the Stability of Reinforced Cylindrical and Conical Shells Subjected to Various Types and Combination of Loads, Section III - General Instability of An Orthotropic Circular Conical Shell Subjected to Hydrostatic Pressure and A Compressive Axial Force," submitted September, 1963, under Contract No. NAS8-5012 to the George C. Marshall Space Flight Center of the National Aeronautics and Space Administration.

CONSTANTS

The constants which appear in equation (36) of the parent report are functions of the physical parameters describing the conical shell. The parameters are as follows: α, the apex angle, α_1, α_2, and α_3 the extensional and shear stiffness coefficients, p, the applied external pressure; Q, the axial load; S_0, the value of S at the base of the cone frustrum; $\gamma_{\theta s}$ and $\gamma_{s\theta}$, the Poissons ratios; h, the wall thickness of the shell; and n, the number of waves in the circumferential direction after buckling has occurred.

Equations (29), (30), and (31) of the parent report can be written in the following form:

$$A_1 F + A_2 F' + A_3 F'' + A_4 G + A_5 G' + A_6 H + A_7 H' = 0 \quad (29)$$

$$B_1 F + B_2 F' + B_3 G + B_4 G' + B_5 G'' + B_6 H = 0 \quad (30)$$

$$\overline{C}_1 F + \overline{C}_2 F' + \overline{C}_3 G + \overline{C}_4 H + \overline{C}_5 H' + \overline{C}_6 H'' + \overline{C}_7 H'''$$

$$+ C_8 H'''' = 0 \quad (31)$$

Where the superscripts appearing on F, G and H denote the derivative with respect to $\overline{S}$. The coefficients A's, B's, and C's are defined as follows:

$$A_1 = (a_2 - a_9 n^2) \frac{1}{\overline{S}} + (a_1 - a_8 n^2), \qquad A_2 = a_3 + a_4 \overline{S}$$

$$A_3 = a_5 + a_6 \overline{S} + a_7 \overline{S}^2 \qquad A_4 = -a_{11} n \frac{1}{\overline{S}} - a_{10} n$$

$$A_5 = -a_{12} n - a_{13} n \overline{S} \qquad A_6 = a_{15} \frac{1}{\overline{S}} + a_{14}$$

$$A_7 = a_{16} + a_{17} \overline{S}$$

$$B_1 = b_2 n \frac{1}{\overline{S}} + b_1 n$$

$$B_2 = b_3 n + b_4 n \overline{S}$$

$$B_3 = (b_6 - b_{13} n^2) \frac{1}{\overline{S}} + (b_5 - b_{12} n^2)$$

$$B_4 = b_7 + b_8 \overline{S}$$

$$B_5 = b_9 + b_{10} \overline{S} + b_{11} \overline{S}^2$$

$$B_6 = b_{15} n \frac{1}{\overline{S}} + b_{14} n$$

$$\overline{C}_1 = c_1 \frac{1}{\overline{S}} + c_2$$

$$\overline{C}_2 = c_3 + c_4 \overline{S}$$

$$\overline{C}_3 = - c_5 n \frac{1}{\overline{S}} - n c_6$$

$$\overline{C}_4 = (c_{20} n^4 - c_{15} n^2) \frac{1}{\overline{S}^2} + c_8 \frac{1}{\overline{S}} + (c_7 - c_{14} n^2)$$

$$\overline{C}_5 = (c_{10} - c_{17} n^2) \frac{1}{\overline{S}^2} + c_9 \overline{S}$$

$$\overline{C}_6 = (c_{13} - c_{19} n^2) \frac{1}{\overline{S}} + c_{11} + c_{12} \overline{S}^2$$

$$\overline{C}_7 = c_{16}$$

$$\overline{C}_8 = c_{18} \overline{S}$$

Equations (29), (30) and (31) can be reduced to two differential equations in terms of functions of F and H only.

$$D_1 F + D_2 F' + D_3 F'' = h_2 \tag{37}$$

$$E_1 F + E_2 F' + E_3 F'' + E_4 F''' = h_3 \tag{38}$$

Where the coefficients D_i, E_i, h_2 and h_3 are defined as follows:

$$D_1 = \overline{C}_1 (A_5 \overline{C}_3' - A_4 \overline{C}_3) + \overline{C}_1' (- A_5 \overline{C}_3) + A_1 \overline{C}_3^2$$

$$D_2 = \overline{C}_2 (A_5 \overline{C}_3' - A_4 \overline{C}_3) + (\overline{C}_1 + \overline{C}_2') (- A_5 \overline{C}_3) + A_2 \overline{C}_3^2$$

$$D_3 = \overline{C}_2 (- A_5 \overline{C}_3) + A_3 \overline{C}_3^2$$

$$h_2 = D_4 H + D_5 H' + D_6 H'' + D_7 H''' + D_8 H'''' + D_9 H'''''$$

$$D_4 = \overline{C}_3^2 (- A_6) + (- \overline{C}_4) (- A_4 \overline{C}_3 + A_5 \overline{C}_3') + (- A_5 \overline{C}_3) (- \overline{C}_4')$$

$$D_5 = \overline{C}_3^2 (- A_7) + (- \overline{C}_5) (- A_4 \overline{C}_3 + A_5 \overline{C}_3') + (- A_5 \overline{C}_3) [- (\overline{C}_4 + \overline{C}_5')]$$

$$D_6 = (- \overline{C}_6) (- A_4 \overline{C}_3 + A_5 \overline{C}_3') + (- A_5 \overline{C}_3) [- (\overline{C}_5 + \overline{C}_6')]$$

$$D_7 = (- \overline{C}_7) (- A_4 \overline{C}_3 + A_5 \overline{C}_3') + (- A_5 \overline{C}_3) [- (\overline{C}_6 + \overline{C}_7')]$$

$$D_8 = (- \overline{C}_8) (- A_4 \overline{C}_3 + A_5 \overline{C}_3') + (- A_5 \overline{C}_3) [- (\overline{C}_7 + \overline{C}_8')]$$

$$D_9 = (- A_5 \overline{C}_3) (- \overline{C}_8)$$

$$h_3 = E_1 F + E_2 F' + E_3 F'' + E_4 F'''$$

$$E_1 = A_1 \; 1 + B_1 (- A_5 \overline{C}_3^2) + \overline{C}_1 \; 2 + C_1'' \; 3$$

$$E_2 = A_2 \; 1 + B_2 (- A_5 \overline{C}_3^2) + \overline{C}_2 \; 2 + (2 \overline{C}_1' + \overline{C}_2'') \; 3$$

$$E_3 = A_3 \; 1 + (\overline{C}_1 + 2 \overline{C}_2') \; 3$$

$$E_4 = \overline{C}_2 \; 3$$

$$h_3 = E_5 H + E_6 H' + E_7 H'' + E_8 H''' + E_9 H'''' + E_{10} H''''' + E_{11} H''''''$$
$$= \mathrm{II}$$

$$1 = - 2 B_5 \overline{C}_3 \overline{C}_3' + B_4 \overline{C}_3^2$$

$$2 = - A_5 B_5 \overline{C}_3'' + 2 A_4 B_5 \overline{C}_3' + A_5 B_3 \overline{C}_3 - A_4 B_4 \overline{C}_3$$

$$3 = A_5 B_5 \overline{C}_3$$

$$E_5 = 4 + \overline{C}_4 \; 5 + 3 \; \overline{C}_4''$$

$$E_6 = 6 + \overline{C}_5 \; 5 + 3 \; (2 \overline{C}_4' + \overline{C}_5)$$

$$E_7 = \overline{C}_6 \, (5) + (3) \, (\overline{C}_4 + 2\,\overline{C}_5{}' + \overline{C}_6{}'')$$

$$E_8 = \overline{C}_7 \, (5) + (3) \, (\overline{C}_5 + 2\,\overline{C}_6{}' + \overline{C}_7{}'')$$

$$E_9 = \overline{C}_8 \, (5) + (3) \, (\overline{C}_6 + 2\,\overline{C}_7{}' + \overline{C}_8{}'')$$

$$E_{10} = (3) \, (\overline{C}_7 + 2\,\overline{C}_8{}')$$

$$E_{11} = (3) \, \overline{C}_8$$

$$(4) = 2\,A_6\,B_5\,\overline{C}_3\,\overline{C}_3{}' + A_5\,B_6\,\overline{C}_3^2 - A_6\,B_4\,\overline{C}_3^2$$

$$(5) = -(2) = A_5\,B_5\,\overline{C}_3'' - 2\,A_4\,B_5\,\overline{C}_3{}' - A_5\,B_3\,\overline{C}_3 + A_4\,B_4\,\overline{C}_3$$

$$(6) = 2\,A_7\,B_5\,\overline{C}_3\,\overline{C}_3{}' - A_7\,B_4\,\overline{C}_3^2$$

Equations (37) and (38) are reduced to one eighth-order Donnel-type differential equations in terms of the function H alone, which is itself a function of $\overline{S}$ or S, the coordinate in the generatrix direction. The equation is the same as equation (36) of the parent report and can be expressed in the equivalent form

$$T_1\,H^8 + T_2\,H^7 + \ldots\ldots\ldots\ldots\ldots\ldots + T_9\,H = 0 \qquad (39)$$

where the T coefficients are defined as follows:

$$T_1 = +\,\overline{K}_4\;\overline{g}_8$$

$$T_2 = +\,\overline{K}_2\,(\overline{f}_7) + \overline{K}_3\;\overline{g}_8 + \overline{K}_4\,(\overline{g}_7 + \overline{g}_8{}')$$

$$T_3 = \overline{K}_1\,\overline{f}_7 + \overline{K}_2\,(\overline{f}_6 + \overline{f}_7{}') + \overline{K}_3\;\overline{g}_7 + \overline{K}_4\,(\overline{g}_6 + \overline{g}_7{}')$$

$$T_4 = \overline{K}_1\,\overline{f}_6 + \overline{K}_2\,(\overline{f}_5 + \overline{f}_6{}') + \overline{K}_3\;\overline{g}_6 + \overline{K}_4\,(\overline{g}_5 + \overline{g}_6{}')$$

$$T_5 = \overline{K}_1 \overline{f}_5 + \overline{K}_2 (\overline{f}_4 + \overline{f}_5') + \overline{K}_3 \overline{g}_5 + \overline{K}_4 (\overline{g}_4 + \overline{g}_5')$$

$$T_6 = \overline{K}_1 \overline{f}_4 + \overline{K}_2 (\overline{f}_3 + \overline{f}_4') + \overline{K}_3 \overline{g}_4 + \overline{K}_4 (\overline{g}_3 + \overline{g}_4')$$

$$T_7 = \overline{K}_1 \overline{f}_3 + \overline{K}_2 (\overline{f}_2 + \overline{f}_3') + \overline{K}_3 \overline{g}_3 + \overline{K}_4 (\overline{g}_2 + \overline{g}_3')$$

$$T_8 = \overline{K}_1 \overline{f}_2 + \overline{K}_2 (\overline{f}_1 + \overline{f}_2') + \overline{K}_3 \overline{g}_2 + \overline{K}_4 (\overline{g}_1 + \overline{g}_2')$$

$$T_9 = \overline{K}_1 \overline{f}_1 + \overline{K}_2 (\overline{f}_1') + \overline{K}_3 \overline{g}_1 + \overline{K}_4 \overline{g}_1'$$

where:

$$\overline{K}_1 = \overline{\lambda}_2' \overline{\lambda}_3 - \overline{\lambda}_2 \overline{\lambda}_3' + \overline{\lambda}_1 \overline{\lambda}_3$$

$$\overline{K}_2 = \overline{\lambda}_2 \overline{\lambda}_3$$

$$\overline{K}_3 = - K_2' \overline{\lambda}_3 + K_2 \overline{\lambda}_3' - K_1 \overline{\lambda}_3$$

$$\overline{K}_4 = - K_2 \overline{\lambda}_3$$

$$\overline{\lambda}_1 = K_2 D_1 - K_1' D_2$$

$$\overline{\lambda}_2 = K_2 D_2 - D_3 (K_1 + K_2')$$

$$\overline{\lambda}_3 = K_1 \overline{\lambda}_2 - K_2 \overline{\lambda}_1$$

$$K_1 = D_1 \overline{D}_3 - \overline{D}_1 D_3$$

$$K_2 = D_2 \overline{D}_3 - \overline{D}_2 D_3$$

$$\overline{D}_1 = D_3 E_1 - E_4 D_1'$$

$$\overline{D}_2 = D_3 E_2 - E_4 (D_1 + D_2')$$

$$\overline{D}_3 = D_3 E_3 - E_4 (D_2 + D_3')$$

$$\overline{E}_1 = D_3 E_5 - E_4 D_4'$$

$$\overline{E}_2 = D_3 E_6 - E_4 (D_4 + D_5')$$

$$\overline{E}_3 = D_3 E_7 - E_4 (D_5 + D_6')$$

$$\overline{E}_4 = D_3 E_8 - E_4 (D_6 + D_7')$$

$$\overline{E}_5 = D_3 E_9 - E_4 (D_7 + D_8')$$

$$\overline{E}_6 = D_3 E_{10} - E_4 (D_8 + D_9')$$

$$\overline{E}_7 = D_3 E_{11} - E_4 (D_9)$$

$$\overline{f}_1 = \overline{D}_3 D_4 - D_3 \overline{E}_1$$

$$\overline{f}_2 = \overline{D}_3 D_5 - D_3 \overline{E}_2$$

$$\overline{f}_3 = \overline{D}_3 D_6 - D_3 \overline{E}_3$$

$$\overline{f}_4 = \overline{D}_3 D_7 - D_3 \overline{E}_4$$

$$\overline{f}_5 = \overline{D}_3 D_8 - D_3 \overline{E}_5$$

$$\overline{f}_6 = \overline{D}_3 D_9 - D_3 \overline{E}_6$$

$$\overline{f}_7 = - D_3 \overline{E}_7$$

$$\overline{g}_1 = K_2 D_4 - D_3 \overline{f}_1'$$

$$\bar{g}_2 = K_2 D_5 - D_3 (\bar{f}_1 + \bar{f}_2')$$

$$\bar{g}_3 = K_2 D_6 - D_3 (\bar{f}_2 + \bar{f}_3')$$

$$\bar{g}_4 = K_2 D_7 - D_3 (\bar{f}_3 + \bar{f}_4')$$

$$\bar{g}_5 = K_2 D_8 - D_3 (\bar{f}_4 + \bar{f}_5')$$

$$\bar{g}_6 = K_2 D_9 - D_3 (\bar{f}_5 + \bar{f}_6')$$

$$\bar{g}_7 = - D_3 (\bar{f}_6 + \bar{f}_7')$$

$$\bar{g}_8 = - D_3 \bar{f}_7$$

The T coefficients calculated in equation (39) are polynomials in $\bar{S}$. A designation which is used to denote all the T_i coefficients and their components is as follows:

$$T_i = t_{i,\,n} \bar{S}^n$$

where $t_{i,\,n}$ denotes the constant coefficient of a particular $\bar{S}$, the subscript i corresponds to the subscript of T_i and n denotes the power of $\bar{S}$. The range of $\bar{S}$ for each order of H is as follows:

$$T_1 = t_{1,\,-31} \bar{S}^{-31} + \ldots\ldots + t_{1,\,0} \bar{S}^0 + t_{1,\,1} \bar{S} + \ldots + t_{1,\,28} \bar{S}^{28}$$

$$T_2 = t_{2,\,-32} \bar{S}^{-32} \ldots\ldots + t_{2,\,27} \bar{S}^{27}$$

$$T_3 = t_{3,\,-33} \bar{S}^{-33} \ldots\ldots + t_{3,\,29} \bar{S}^{29}$$

$$T_4 = t_{4,\,-34} \bar{S}^{-34} \ldots\ldots + t_{4,\,28} \bar{S}^{28}$$

$$T_5 = t_{5,\,-35} \bar{S}^{-35} \ldots\ldots + t_{5,\,27} \bar{S}^{27}$$

$$T_6 = t_{6,-36}\,\bar{S}^{-36} \cdots\cdots\cdots\cdots + t_{6,26}\,\bar{S}^{26}$$

$$T_7 = t_{7,-37}\,\bar{S}^{-37} \cdots\cdots\cdots\cdots + t_{7,25}\,\bar{S}^{25}$$

$$T_8 = t_{8,-38}\,\bar{S}^{38} \cdots\cdots\cdots\cdots + t_{8,24}\,\bar{S}^{24}$$

$$T_9 = t_{9,-39}\,\bar{S}^{39} \cdots\cdots\cdots\cdots + t_{9,23}\,\bar{S}^{23}$$

The powers of $\bar{S}$, for any particular T_i, increase uniformly until the highest power is reached. This fact is indicated above by the dashes between the term of the lowest order and the term of the highest order.

FORTRAN II COMPUTER PROGRAM

GENERAL

The following FORTRAN II computer program has been developed to calculate the T_i coefficients of equation (39) of this report from data consisting of the parameters α, D_1, D_2, D_3, α_1, α_2, α_3, p, Q, S_0, $\gamma_{\theta s}$, h and n. The manner in which these data words are read into the computer is presented under FORTRAN II COMPUTER PROGRAM INPUT AND OUTPUT INFORMATION of this report. The values of the coefficients calculated in intermediate steps are printed as they are computed so that they may be checked at any point of progression of the program. The appropriate heading is printed immediately above each coefficient printout in order to specify the coefficients being printed. Full details on printout information is found under FORTRAN II COMPUTER PROGRAM INPUT AND OUTPUT INFORMATION of this report.

FORTRAN II LEGEND FOR INPUT DATA

The FORTRAN II designation for input data words is as follows:

Input parameter	FORTRAN II name
α	AL
D_1	DE1
D_2	DE2
D_3	DE3
α_1	AL1
α_2	AL2
α_3	AL3
p	P
Q	Q

S	SO
$\gamma_{\theta s}$	GAM
h	H
n	A

FORTRAN II LEGEND FOR COMPUTED VARIABLES

Due to the restrictions on variable names in the FORTRAN II language, many of the coefficient names were reassigned. The following table gives a complete list of the coefficients with their corresponding FORTRAN II designation. In every case, the subscript (1) corresponds to the lowest power of $\overline{S}$, the highest subscript corresponds to the highest power of $\overline{S}$. For example, D1(1) is the FORTRAN II name for $d_{1,-3}$, T9(1) is the FORTRAN II name for $t_{9,-39}$, etc. The dimension of each variable is its total number of subscripts.

Coefficient	FORTRAN II Variable Name	Lowest Power of $\overline{S}$	Highest Power of $\overline{S}$	Dimension
D_1	D1	-3	0	4
D_2	D2	-2	1	4
D_3	D3	-2	2	5
D_4	D4	-5	0	6
D_5	D5	-4	1	6
D_6	D6	-3	2	6
D_7	D7	-2	3	6

D_8	D8	-1	1	3
D_9	D9	0	2	3
E_1	E1	-4	1	6
E_2	E2	-3	2	6
E_3	E3	-3	3	7
E_4	E4	-1	4	6
E_5	E5	-6	1	8
E_6	E6	-5	2	8
E_7	E7	-4	3	8
E_8	E8	-3	4	8
E_9	E9	-2	5	8
E_{10}	E10	-1	3	5
E_{11}	E11	0	4	5
$\overline{D}_1$	D1B	-6	3	10
$\overline{D}_2$	D2B	-5	4	10
$\overline{D}_3$	D3B	-5	5	11

$\overline{E}_1$	E1B	-8	3	12
$\overline{E}_2$	E2B	-7	4	12
$\overline{E}_3$	E3B	-6	5	12
$\overline{E}_4$	E4B	-5	6	12
E_5	E5B	-4	7	12
E_6	E6B	-3	5	9
E_7	E7B	-2	6	9
$\overline{f}_1$	F1B	-10	5	16
$\overline{f}_2$	F2B	-9	6	16
$\overline{f}_3$	F3B	-8	7	16
$\overline{f}_4$	F4B	-7	8	16
$\overline{f}_5$	F5B	-6	9	16
$\overline{f}_6$	F6B	-5	7	13
$\overline{f}_7$	F7B	-4	8	13
K_1	R1	-8	5	14
K_2	R2	-7	6	14

$\bar{\lambda}_1$	Y1	-11	6	18
$\bar{\lambda}_2$	Y2	-10	7	18
$\bar{\lambda}_3$	Y3	-18	12	31
$\bar{K}_1$	R1B	-29	18	48
$\bar{K}_2$	R2B	-28	19	48
$\bar{K}_3$	R3B	-26	17	44
$\bar{K}_4$	R4B	-25	18	44
$\bar{g}_1$	G1B	-13	6	20
$\bar{g}_2$	G2B	-12	7	20
$\bar{g}_3$	G3B	-11	8	20
$\bar{g}_4$	G4B	-10	9	20
$\bar{g}_5$	G5B	-9	10	20
$\bar{g}_6$	G6B	-8	11	20
$\bar{g}_7$	G7B	-7	9	17
$\bar{g}_8$	G8B	-6	10	17

FORTRAN II LEGEND FOR INTERMEDIATE TERMS

In calculating certain coefficients, it was necessary to utilize some intermediate terms. The intermediate terms are specified as follows:

Intermediate Term	FORTRAN II Name	Lowest Power of $\bar{S}$	Highest Power of $\bar{S}$	Dimension
$D_3 E_1$	D3E1	-6	3	10
$E_4 D_1'$	E4D1P	-5	3	9
$D_3 E_2$	D3E2	-5	4	10
$D_1 + D_2'$	D12P	-3	0	4
$E_4 (D_1 + D_2')$	E412P	-4	4	9
$D_3 E_3$	D3E3	-5	5	11
$D_2 + D_3'$	D23P	-3	1	5
$E_4 (D_2 + D_3')$	E423P	-4	5	10
$D_3 E_5$	D3E5	-8	3	12
$E_4 D_4'$	E4D4P	-7	3	11
$D_3 E_6$	D3E6	-7	4	12
$D_4 + D_5'$	D45P	-5	0	6
$E_4 (D_4 + D_5')$	E445P	-6	4	11

$D_3\ E_7$	D3E7	-6	5	12
$D_5 + D_6'$	D56P	-4	1	6
$E_4\ (D_5 + D_6')$	E456P	-5	5	11
$D_3\ E_8$	D3E8	-5	6	12
$D_6 + D_7'$	D67P	-3	2	6
$E_4\ (D_6 + D_7')$	E467P	-4	6	11
$D_3\ E_9$	D3E9	-4	7	12
$D_7 + D_8'$	D78P	-2	3	6
$E_4\ (D_7 + D_8')$	E478P	-3	7	11
$D_3\ E_{10}$	D3E10	-3	5	9
$D_8 + D_9'$	D89P	-1	1	3
$E_4\ (D_8 + D_9')$	E489P	-2	5	8
$D_3\ E_{11}$	D3E11	-2	6	9
$E_4\ D_9$	E4D9	-1	6	8
$\overline{D}_3\ D_4$	D3BD4	-10	5	16
$D_3\ \overline{E}_1$	D3E1B	-10	5	16

$\overline{D}_3$ D_5	D3BD5	-9	6	16
D_3 $\overline{E}_2$	D3E2B	-9	6	16
$\overline{D}_3$ D_6	D3BD6	-8	7	16
D_3 $\overline{E}_3$	D3E3B	-8	7	16
$\overline{D}_3$ D_7	D3BD7	-7	8	16
D_3 $\overline{E}_4$	D3E4B	-7	8	16
$\overline{D}_3$ D_8	D3BD8	-6	6	13
D_3 $\overline{E}_5$	D3E5B	-6	9	16
$\overline{D}_3$ D_9	D3BD9	-5	7	13
D_3 $\overline{E}_6$	D3E6B	-5	7	13
D_3 $\overline{E}_7$	D3E7B	-4	8	13
D_1 $\overline{D}_3$	D1D3B	-8	5	14
$\overline{D}_1$ D_3	D1BD3	-8	5	14
D_2 $\overline{D}_3$	D2D3B	-7	6	14
$\overline{D}_2$ D_3	D2BD3	-7	6	14
K_2 D_1	R2D1	-10	6	17

$K_1' D_2$	R1PD2	-11	5	17
$K_2 D_2$	R2D2	-9	7	17
$(K_1 + K_2')$	R12	-8	5	14
$D_3 (K_1 + K_2')$	D312K	-10	7	18
$K_1 \bar{\lambda}_2$	R1Y2	-18	12	31
$K_2 \bar{\lambda}_1$	R2Y1	-18	12	31
$\bar{\lambda}_2' \lambda_3$	Y2PY3	-29	18	48
$\bar{\lambda}_2 \bar{\lambda}_3'$	Y2Y3P	-29	18	48
$\bar{\lambda}_1 \bar{\lambda}_3$	Y1Y3	-29	18	48
$K_2' \bar{\lambda}_3$	R2PY3	-26	17	44
$K_2 \bar{\lambda}_3'$	R2Y3P	-26	17	44
$K_1 \bar{\lambda}_3$	R1Y3	-26	17	44
$K_2 D_4$	R2D4	-12	6	19
$D_3 \bar{f}_1'$	D3F1P	-13	6	20
$K_2 D_5$	R2D5	-11	7	19
$(\bar{f}_1 + \bar{f}_2')$	F12	-10	5	16

$D_3\ (\bar{f}_1 + \bar{f}_2')$	D312	-12	7	20
$K_2\ D_6$	R2D6	-10	8	19
$(\bar{f}_2 + \bar{f}_3')$	F23	-9	6	16
$D_3\ (\bar{f}_2 + \bar{f}_3')$	D323	-11	8	20
$K_2\ D_7$	R2D7	-9	9	19
$(\bar{f}_3 + \bar{f}_4')$	F34	-8	7	16
$D_3\ (\bar{f}_3 + \bar{f}_4')$	D334	-10	9	20
$K_2\ D_8$	R2D8	-8	7	16
$(\bar{f}_4 + \bar{f}_5')$	F45	-7	8	16
$D_3\ (f_4 + \bar{f}_5')$	D345	-9	10	20
$K_2\ D_9$	R2D9	-7	8	16
$(\bar{f}_5 + \bar{f}_6')$	F56	-6	9	16
$D_3\ (\bar{f}_5 + \bar{f}_6')$	D356	-8	11	20
$(\bar{f}_6 + \bar{f}_7')$	F67	-5	7	13
$\bar{K}_2\ \bar{f}_7$	R2F7	-32	27	60
$\bar{K}_3\ \bar{g}_8$	R3G8	-32	27	60

$(\overline{g}_7 + \overline{g}_8')$	G78	-7	9	17
$\overline{K}_4\ (\overline{g}_7 + \overline{g}_8')$	R478	-32	27	60
$\overline{K}_1\ \overline{f}_7$	R1F7	-33	26	60
$\overline{K}_2\ (\overline{f}_6\ +$ [illegible]	R267Z	-33	26	60
$K_3\ \overline{g}_7$	R3G7	-33	26	60
$(\overline{g}_6 + \overline{g}_7')$	G67	-8	11	20
$\overline{K}_4\ (\overline{g}_6 + \overline{g}_7')$	R467G	-33	29	63
$\overline{K}_1\ \overline{f}_6$	R1F6	-34	25	60
$\overline{K}_2\ (\overline{f}_5 + \overline{f}_6')$	R256Z	-34	28	63
$\overline{K}_3\ \overline{g}_6$	R3G6	-34	28	63
$(\overline{g}_5 + \overline{g}_6')$	G56	-9	10	20
$\overline{K}_4\ (\overline{g}_5 + \overline{g}_6')$	R456G	-34	28	63
$\overline{K}_1\ \overline{f}_5$	R1F5	-35	27	63
$\overline{K}_2\ (\overline{f}_4 + \overline{f}_5')$	R245Z	-35	27	63
$\overline{K}_3\ \overline{g}_5$	R3G5	-35	27	63
$(\overline{g}_4 + \overline{g}_5')$	G45	-10	9	20
$\overline{K}_4\ (\overline{g}_4 + \overline{g}_5')$	R445G	-35	27	63
$\overline{K}_1\ \overline{f}_4$	R1F4	-36	26	63
$\overline{K}_2\ (\overline{f}_3 + \overline{f}_4')$	R234Z	-36	26	63
$\overline{K}_3\ \overline{g}_4$	R3G4	-36	26	63
$(\overline{g}_3 + \overline{g}_4')$	G34	-11	8	20

$\overline{K}_4 (\overline{g}_3 + \overline{g}_4')$	R434G	-36	26	63
$\overline{K}_1 \overline{f}_3$	R1F3	-37	25	63
$\overline{K}_2 (\overline{f}_2 + \overline{f}_3')$	R223Z	-37	25	63
$\overline{K}_3 \overline{g}_3$	R3G3	-37	25	63
$(\overline{g}_2 + \overline{g}_3')$	G23	-12	7	20
$\overline{K}_4 (\overline{g}_2 + \overline{g}_3')$	R423G	-37	25	63
$\overline{K}_1 \overline{f}_2$	R1F2	-38	24	63
$\overline{K}_2 (\overline{f}_1 + \overline{f}_2')$	R212Z	-38	24	63
$\overline{K}_3 \overline{g}_2$	R3G2	-38	24	63
$(\overline{g}_1 + \overline{g}_2')$	G12	-13	6	20
$\overline{K}_4 (\overline{g}_1 + \overline{g}_2')$	R412G	-38	24	63
$\overline{K}_1 \overline{f}_1$	R1F1	-39	23	63
$\overline{K}_2 \overline{f}_1'$	R2F1P	-39	23	63
$\overline{K}_3 \overline{g}_1$	R3G1	-39	23	63
$\overline{K}_4 \overline{g}_1'$	R4G1P	-39	23	63

FORTRAN II COMPUTER PROGRAM INPUT AND OUTPUT INFORMATION

Input . The input data which must be punched on cards are the constants α , D_1 , D_2 , D_3 , α_1 , α_2 , α_3 , P, Q, S, $\gamma_{\theta S}$, h, n. This data is punched on four cards.

The first data card will contain α in spaces 1 through 18, D_1 in spaces 19 through 36, D_2 in spaces 37 through 54, and D_3 in spaces 55 through 72.

The second data card will contain α_1 in spaces 1 through 18, α_2 in spaces 19 through 36, α_3 in spaces 37 through 54, and P in spaces 55 through 72.

The third data card will contain Q in spaces 1 through 18, S in spaces 19 through 36, $\gamma_{\theta S}$ in spaces 37 through 54, and h in spaces 55 through 72.

The fourth data card will contain the value of n in spaces 1 through 18.

The format for the values to be punched in their above assigned spaces is as follows: The value of the constant is preceded by the algebraic sign of plus or minus. A decimal point is punched between the first and second character of the value. The value is followed by an exponent sign of plus or minus. Following the exponent sign is the power of the exponent. This format is shown in the example below.

Example:

α = 22 . 1

algebraic sign — exponent

Punch Format for α: + 2. 21 + 01

decimal point — exponent sign

α = -263

Punch Format for α: - 2. 63 + 02

All the constants should be punched according to the example format and in the extreme right of their allotted spaces.

Output. A descriptive title is printed before each section of the program printout. All of these values are printed out in floating point numbers. This format explanation is the same as the punch format explanation above.

The first descriptive title is, "A 1 through A 17." This means that the following values in this section are the values for A 1 through A 17. If the first value was - 0. 2336111 E + 01, this would be written in fixed point arithmetic as - 2. 336111. The following sections of the printout would be interpreted the same as the above section.

FORTRAN II SOURCE PROGRAM

The program which defines the operations which the computer is to do, and which is written by the programmer in the FORTRAN II language is called the FORTRAN II Source Program.

An occasional comment and the inclusion of defining equations to facilitate the correlation of program variables to report variables will occur at certain intervals in the source program. This is to be regarded as supplemental information only and not an integral part of the FORTRAN II instructions. With this in mind and with reference to FORTRAN II Legends presented earlier in this report, the reader should have little difficulty in understanding the following source program.

```
C       INSTABILITY ØF AN ØRTHØTRØPIC CIRCULAR
C       CØNICAL SHELL SUBJECTED TØ HYDRØSTATIC
C       PRESSURE AND A CØMPRESSIVE AXIAL FØRCE
        DIMENSIØN D1(4), D2(4), D3(5), D4(6), D5(6), D6(6), D7(6), D8(3),
       1D9(3), E1(6), E2(6), E3(7), E4(6), E5(8) , E6(8), E7(8), E8(8), E9(
       28), E10(5), E11(5), D1P(4), D2P(4), D3P(5), D4P(6), D5P(6), D6P(6)
       3, D7P(6), D8P(3), D9P(3), D1B(10), D2B(10), D3B(11), E1B(12), E2B(
       412), E3B(12), E4B(12), E5B(12), E6B(9), E7B(9), F1B(16), F2B(16),
       5F3B(16), F4B(16), F5B(16), F6B(13), F7B(13), F1BP(16), F2BP(16), F3
       6BP(16), F4BP(16), F5BP(16), F6BP(13), F7BP(13), R1(14), R2(14), R1
       7P(14), R2P(14), Y1(18), Y2(18), Y3(31), R1B(48), R2B(48), R3B(44),
       8 R4B(44), G1B(20), G2B(20), G3B(20), G4B(20), G5B(20), G6B(20), G7
       9B(17), G8B(17)
        DIMENSIØN G1BP(20), G2BP(20), G3BP(20), G4BP(20), G5BP(20), G6BP(2
       10), G7BP(17), G8BP(17), T1(60), T2(60), T3(63), T4(63), T5(63), T6
       2(63), T7(63), T8(63), T9(63), D3E1(10), E4D1P(10), D3E2(10), D12P(
       34), E412P(9), D3E3(11), D23P(5), E423P(10), D3E5(12), E4D4P(11), D
       43E6(12), D45P(6), E445P(11), D3E7(12), D56P(6), E456P(11), E3E8(12
       5), D67P(6), E467P(11), D3E9(12), D78P(6), E478P(11), D3E10(9), D89
       6P(3), E489P(8), D3E11(9), E4D9(8) , D3BD4(16), D3E1B(16), D3BD5(16)
       7, D3E2B(16), D3BD6(16), D3E3B(16), D3BD7(16), D3E4B(16), D3BD8(13)
       8, D3E5B(16), D3BD9(13), D3E6B(13), D3E7B(13), D1D3B(14), D1BD3(14)
       9, D2D3B(14), D2BD3(14), R2D1(17), R1PD2(17), R2D2(17), R12(14)
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      DIMENSIØN D312K(48), R1Y2(31), R2Y1(31), Y2PY3(48), Y2Y3P(48), Y1Y
     13(18), R2PY3(44), R2Y3P(44), R1Y3(44), R2D4(19), D3F1P(20), R2D5(1
     29), F12(16), D312(20), R2D6(19), F23(16), D323(20), R2D7(19), F34(
     316), D334(20), R2D8(16), F45(16), D345(20), R2D9(16), F56(16), D35
     46(20), F67(13), R2F7(60), R3G8(60), G78(17), R478(60), R1F7(60), R
     5267Z(60), R3G7(60), G67(20), R467G(63), R1F6(60), R256Z(63), R3G6(
     663), G56(20), R456G(63), R1F5(63), R245Z(63), R3G5(63), G45(20), R
     7445G(63), R1F4(63), R234Z(63), R3G4(63), G34(20), R434G(63), R1F3(
     863), R223Z(63), R3G3(63), G23(20), R423G(63), R1F2(63), R212Z(63),
     9 R3G2(63), G12(20), R412G(63), R1F1(63), R2F1P(63), R3G1(63)
      DIMENSIØN R4G1P(63), Y1P(18), Y2P(18), Y3P(31), R245(16), R2F1(16)
      CØMMØN SAL, CAL, SCAL, TAL, A1, A2, A3, A4, A5, A6, A7, A8, A9, A1
     10, A11, A12, A13, A14, A15, A16, A17, B1, B2, B3, B4, B5, B6, B7,
     2B8, B9, B10, B11, B12, B13, B14, B15, C1, C2, C3, C4, C5, C6, C7,
     3C8, C9, C10, C11, C12, C13, C14, C15, C16, C17, C18, C19, C20, ASQ
     4, A4H, ST1, ST2, ST3, ST4, ST5, ST6, ST7, ST8, ST9, ST10, ST11, ST
     512, ST13, ST14, ST15, ST16, ST17, ST18, ST19, ST20, ST21, ST22, ST
     623, ST24, ST25, ST26, ST27, ST28, ST29, ST30, ST31, ST32, ST33, ST
     734, ST35, ST36, ST37, ST38, ST39, ST40, ST41, ST42, ST43, ST44, ST
     845, D1, D2, D3, D4, D5, D6, D7, D8, D9, E1, E2, E3, E4, E5, E6, E7
     9, E8, E9, E10, E11, D1P, D2P, D3P, D4P, D5P, D6P, D7P, D8P, D9P
      CØMMØN D1B, D2B, D3B, E1B, E2B, E3B, E4B, E5B, E6B, E7B, F1B, F2B,
     1F3B, F4B, F5B, F6B, F7B, F1BP, F2BP, F3BP, F4BP, F5BP, F6BP, F7BP,
     2 R1, R2, R1P, R2P, Y1, Y2, Y3, R1B, R2B, R3B, R4B, G1B, G2B, G3B,
     3G4B, G5B, G6B, G7B, G8B, G1BP, G2BP, G3BP, G4BP, G5BP, G6BP, G7BP,
     4 G8BP, T1, T2, T3, T4, T5, T6, T7, T8, T9, D3E1, E4D1P, D3E2, D12P
     5, E412P, D3E3, D23P, E423P, D3E5, E4D4P, D3E6, D45P, E445P, D3E7,
     6D56P, E456P, D3E8, D67P, E467P, D3E9, D78P, E478P, D3E10, D89P, E4
     789P, D3E11, E4D9, D3BD4, D3E1B, D3BD5, D3E2B, D3BD6, D3E3B, D3BD7,
     8 D3E4B, D3BD8, D3E5B, D3BD9, D3E6B, D3E7B, D1D3B, D1BD3, D2D3B, D2
     9BD3, R2D1, R1PD2, R2D2, R12, D312K, R1Y2, R2Y1, Y2PY3, Y2Y3P, Y1Y3
      CØMMØN R2PY3, R2Y3P, R1Y3, R2D4, D3F1P, R2D5, F12, D312, R2D6, F23,
```

```
1 D323, R2D7, F34, D334, R2D8, F45, D345, R2D9, F56, D356, F67, R2F
27, R3G8, G78, R478, R1F7, R267Z, R3G7, G67, R467G, R1F6, R256Z, R3
3G6, G56, R456G, R1F5, R245Z, R3G5, G45, R445G, R1F4, R234Z, R3G4,
4G34, R434G, R1F3, R223Z, R3G3, G23, R423G, R1F2, R212Z, R3G2, G12,
5 R412G, R1F1, R2F1P, R3G1, R4G1P
 READ INPUT TAPE 5, 1, AL, DE1, DE2, DE3, AL1, AL2, AL3, P, Q, SØ,
1GAM, H, A
 WRITE ØUTPUT TAPE 6, 402, AL, DE1, DE2, DE3, AL1, AL2, AL3, P, Q,
1SØ, GAM, H, A
 SAL = SINF(AL)
 CAL = CØSF(AL)
 SCAL = ((SINF(AL))*(SINF(AL))) ((CØSF (AL))
 TAL = SAL/CAL
 A1 = -(3.0*P*SØ*SAL * *2)/(2.0*AL1*CAL)
 A2 = AL2*SAL/AL1
 A3 = -SAL
 A4 = (2.0*P*SØ*SCAL)/AL1
 A5 = Q/(6.2831852*SØ*AL1*CAL)
 A6 = A3
 A7 = (P*SØ*SCAL)/AL1
 A8 = (3.0*P*SØ)/(2.0*AL1*CAL)
 A9 = -(AL3/(AL1*SAL))
 A10 = -((3.0*P*SØ*TAL)/(2.0*AL1))
 A11 = (AL2 + AL3)/AL1
 A12 = -((AL1*GAM) + AL3)/AL1
 A13 = (P*SØ*TAL)/(2.0*AL1)
 A14 = (3.0*P*SØ*SAL)/(2.0*AL1)
 A15 = -(AL2*CAL)/AL1
 A16 = GAM*CAL
```

A17 = -(P*SØ*SAL)/AL1

B1 = (2.0*P*SØ*TAL)/AL1

B2 = -(AL2 + AL3)/AL1

B3 = -((GAM*AL1) + AL3)/AL1

B4 = (P*SØ*TAL)/(2.0*AL1)

B5 = -(2.0*P*SØ*SCAL)/AL1

B6 = (AL3*SAL)/AL1

B7 = -(AL3*SAL)/AL1

B8 = (2.0*SØ*P*SCAL)/AL1

B9 = Q/(6.2831852*SØ*AL1*CAL)

B10 = -(AL3*SAL)/AL1

B11 = (P*SØ*SCAL)/AL1

B12 = (3.0*P*SØ)/(2.0*AL1*CAL)

B13 = -(AL2)/(AL1*SAL)

B14 = -(2.0*P*SØ)/AL1

B15 = (AL2)/(TAL*AL1)

C1 = -(AL2*CAL)/AL1

C2 = (5.0*P*SØ*SAL)/(2.0*AL1)

C3 = -(GAM*CAL)

C4 = (P*SØ*SAL)/AL1

C5 = -(AL2)/(TAL*AL1)

C6 = (2.0*SØ*P)/AL1

C7 = ((3.0*P*SØ*CAL*CAL) + (2.0*P*SØ*SAL*SAL))/(2.0*AL1*CAL)

C8 = (AL2*CAL*CAL) / (AL1*SAL)

```
C9 = (2. 0*P*SØ*SCAL)/AL1
C10 = (H*H*DE2*SAL)/(12. 0*SØ*SØ*DE1)
C11 = B9
C12 = C9/2. 0
C13 = -C10
C14 = B12
C15 = ((H*H*GAM*DE1)+(H*H*DE2)+(2. 0*H*H*DE3))/(6. 0*SØ*SØ*DE1*S
1AL)
C16 = (2. 0*H*H*SAL)/(12. 0*SØ*SØ)
C17 = -((GAM*H*H*DE1) + (2. 0*H*H*DE3))/(6. 0*SØ*SØ*DE1*SAL)
C18 = (H*H*SAL)/(12. 0*SØ*SØ)
C19 = -C17
C20 = (H*H*DE2)/(12. 0*SØ*SØ*DE1*(SAL**3))
WRITE ØUTPUT TAPE 6, 404
WRITE ØUTPUT TAPE 6, 405, A1, A2, A3, A4, A5, A6, A7, A8, A9, A10,
1 A11, A12, A13, A14, A15, A16, A17
WRITE ØUTPUT TAPE 6, 407
WRITE ØUTPUT TAPE 6, 405, B1, B2, B3, B4, B5, B6, B7, B8, B9, B10,
1 B11, B12, B13, B14, B15
WRITE ØUTPUT TAPE 6, 408
WRITE ØUTPUT TAPE 6, 405, C1, C2, C3, C4, C5, C6, C7, C8, C9, C10,
1 C11, C12, C13, C14, C15, C16, C17, C18, C19, C20
ASQ = A*A
A4H = ASQ*ASQ
```

ST1 = (C20*ASQ) - C15

ST2 = C10 - (ASQ*C17)

ST3 = C13 - (ASQ*C19)

ST4 = C5*C5

ST5 = (C14*ASQ) - C7

ST6 = (C5*A10) + (C6*A11)

ST7 = (C5*A13) + (C6*A12)

ST8 = A2 - (A9*ASQ)

ST9 = 2. 0*C5*C6

ST10 = C2*C5

ST11 = C1*C6

ST12 = A1 - (A8*ASQ)

ST13 = C6*C6

ST14 = C3*C5

ST15 = C5*A12

ST16 = C5*A11

ST17 = C5*A13

ST18 = C6*A13

Since the appearance of the program variables ST1, ST2, etc. will undoubtedly appear confusing to the unsuspecting reader an explanation by example will be given.

Consider the variable **D1** which is defined earlier in the report as follows:

$$D_1 = \overline{C}_1 (A_5 \overline{C}^{1} - A_4 \overline{C}_3) + \overline{C}_i (-A_5 \overline{C}_3) + A_1 \overline{C}_3^2$$

Substituting for the terms in terms of previously defined elements, the equation becomes:

$$D_1 = \left(c_1 \frac{1}{\bar{s}} + c_2 \right) \left[\left(- a_{12}\, n - a_{13}\, n\, \bar{S} \right) \frac{c_5\, n}{\bar{s}^2} - \left(- a_{11}\, n \frac{1}{\bar{s}} - a_{10}\, n \right) \left(-c_5\, n \frac{1}{\bar{s}} - n\, c_6 \right) \right]$$

$$+ \left(- \frac{c_1}{\bar{s}^2} \right) \left(a_{12}\, n + a_{13}\, n\, \bar{s} \right) \left(- c_5\, n \frac{1}{\bar{s}} - n\, c_6 \right) + \left[\left(a_2 - a_9\, n^2 \right) \frac{1}{\bar{s}}\right.$$

$$\left. + \left(a_1 - a_8\, n^2 \right) \right] \left(- c_5\, n \frac{1}{\bar{s}} - n\, c_6 \right)^2$$

A complete algebraic expansion will result in an equation consisting of 28 terms as follows:

$$D_1 = - \frac{c_1\, a_{12}\, c_5\, n_2}{\bar{s}^3} - \frac{c_2\, a_{12}\, c_5\, n^2}{\bar{s}^2} - \frac{c_1\, a_{13}\, c_5\, n^2}{\bar{s}^2} - \frac{c_2\, a_{13}\, c_5\, n^2}{\bar{s}}$$

$$- \frac{c_5\, n^2\, c_1\, a_{11}}{\bar{s}^3} - \frac{c_5\, n^2\, c_2\, a_{11}}{\bar{s}^2} - \frac{c_5\, n^2\, c_1\, a_{10}}{\bar{s}^2} - \frac{c_5\, n^2\, c_2\, a_{10}}{\bar{s}}$$

$$- \frac{c_1\, a_{11}\, n^2\, c_6}{\bar{s}^2} - \frac{c_2\, a_{11}\, n^2\, c_6}{\bar{s}} - \frac{c_1\, a_{10}\, n^2\, c_6}{\bar{s}} - c_2\, a_{10}\, n^2\, c_6$$

$$+ \frac{a_{12}\, n^2\, c_5\, c_1}{\bar{s}^3} + \frac{c_1\, a_{13}\, n^2\, c_5}{\bar{s}^2} + \frac{c_1\, a_{12}\, n^2\, c_6}{\bar{s}^2} + \frac{c_1\, a_{13}\, n^2\, c_6}{\bar{s}}$$

$$+ \frac{a_2\, c_5{}^2\, n^2}{\bar{s}^3} + \frac{2\, c_5\, n^2\, c_6\, a_2}{\bar{s}^2} + \frac{a_2\, c_6{}^2\, n^2}{\bar{s}} - \frac{a_9\, n^2\, c_5{}^2\, n^2}{\bar{s}^3}$$

$$- \frac{2\, a_9\, n^2\, c_5\, n^2 c_6}{\bar{s}^2} - \frac{a_9\, n^4\, c_6{}^2}{\bar{s}} + \frac{a_1\, c_5{}^2\, n^2}{\bar{s}^3} + \frac{2\, a_1\, c_6\, c_5\, n^2}{\bar{s}}$$

$$+ a_1\, c_6{}^2\, n^2 - \frac{a_8\, n^4\, c_5{}^2}{\bar{s}^2} - \frac{2\, a_8\, n^4\, c_6\, c_5}{\bar{s}} - a_8 n^4\, c_6{}^2$$

A grouping of coefficients will be:

Term	Coefficient
$\bar{s}^{-3}$	$- c_1 a_{12} c_5 n^2 - c_5 n^2 c_1 a_{11} + a_{12} n^2 c_5 c_1 + a_2 c_5^2 n^2 - a_9 n^4 c_5^2$
$\bar{s}^{-2}$	$- c_2 a_{12} n^2 c_5 - c_5 n^2 c_2 a_{11} - c_5 n^2 c_1 a_{10} - c_1 a_{11} n^2 c_6 + c_1 a_{12} n^2 c_6 + 2 c_5 n^2 c_6 a_2 - 2 a_9 n^4 c_5 c_6 + a_1 c_5^2 n^2 - a_8 n^4 c_5^2$
$\bar{s}^{-1}$	$- c_5 n^2 c_2 a_{10} - c_2 a_{11} n^2 c_6 - c_1 a_{10} n^2 c_6 + c_1 a_{13} n^2 c_6 + a_2 c_6^2 n^2 - a_9 n^4 c_6^2 - 2 a_8 n^4 c_6 c_5 + 2 c_5 c_6 a_1 n^2 - c_2 c_5 a_{13} n^2$
$\bar{s}^{0}$	$- c_2 a_{10} n^2 c_6 + a_1 c_6^2 n^2 - a_8 n^4 c_6^2$

As mentioned earlier the subscript (1) corresponds to the lowest power of $\bar{s}$; the highest subscript corresponds to the highest power of $\bar{s}$. With this notation in mind it is clear that D1(1) is the coefficient of $\bar{s}^{-3}$, D1(2) is the coefficient of $\bar{s}^{-2}$, etc. Referring to the FORTRAN II source program the variables are written:

```
D1(1) = (ST4*ASQ*(A2 - (A9*ASQ))) - C1*C5*A11*ASQ )
D1(2) = ASQ*(((ST4*(A1 - (A8*ASQ)))) + (ST9*ST8)
          - (C1*ST6) - (ST10*A12) + (ST11*A12))
D1(3) = ASQ*((C6*C6*ST8) + (ST9*ST12) - (ST11*A10)
          - (C2*ST6) - (ST10*A13) + (ST11*A13))
D1(4) = ASQ*((ST13*ST12) - (C2*C6*A10))
```

Substituting for the ST - terms as defined in the SOURCE PROGRAM and expanding, it is seen that the terms are as follows:

D1(1) - coefficient of $\bar{s}^{-3}$

D1(2) - coefficient of $\bar{s}^{-2}$

D1(3) - coefficient of $\bar{s}^{-1}$

D1(4) - coefficient of $\bar{s}^{-0}$

With the aforementioned example for the solution of the term D_1, it should be intuitively obvious to the reader that the ST - terms are characteristic terms employed to facilitate the solution of equations defining D1(1) through D9(3).

The reader may now proceed with the SOURCE PROGRAM.

```
D1(1) = (ST4*ASQ*(A2 - (A9*ASQ))) - (C1*C5*A11*ASQ)

D1(2) = ASQ*(((ST4*(A1 - (A8*ASQ)))) + (ST9*ST8) - (C1*ST6) - (ST1
10*A11) - (ST10*A12) + (ST11*A12))

D1(3) = ASQ*((C6*C6*ST8) + (ST9*ST12) - (ST11*A10) - (C2*ST6) - (S
1T10*A13) + (ST11*A13))

D1(4) = ASQ*((ST13*ST12) - (C2*C6*A10))

D2(1) = ASQ*((ST4*A3) - (ST14*A11) - (C1*ST15) - (ST14*A12))

D2(2) = ASQ*((ST9*A3) + (ST4*A4) - (C3*ST6) - (C4*C5*A11) - (C1*S
1T7) - (ST10*A12) - (C4*ST15) - (ST14*A13) - (C4*ST15))

D2(3) = ASQ*((ST9*A4) + (ST13*A3) - (C3*C6*A10) - (C4*ST6) - (ST11
1*A13) - (C2*ST7) - (C4*ST7) - (C4*C5*A13))

D2(4) = ASQ*((ST13*A4) - (C4*C6*A10) - (C2*C6*A13) - (C4*C6*A13))
```

```
D3(1) = ST4*ASQ*A5

D3(2) = ASQ*((ST4*A6)+(ST9*A5) - (ST14*A12))

D3(3) = ASQ*((ST9*A6) + (ST4*A7) + (ST13*A5) - (C3*ST7) + (C4*ST15
1))

D3(4) = ASQ*((ST9*A7) + (ST13*A6) - (C3*C6*A13) - (C4*ST7))

D3(5) = ASQ*((ST13*A7) - (C4*C6*A13))

D4(1) = A4H*((ST16*ST1) + (ST15*ST1) - (3.0*ST15*ST1))

D4(2) = A4H*ST1*(ST6 + ST17 - (3.0*ST7))

D4(3) = ASQ*((-ST4*A15) + (ASQ*C6*A10*ST1) + (ST16*C8) + (ST15* C8)
1 - (3.0*ASQ*C6*A13*ST1) - (ST15* C8))

D4(4) = ASQ*((-ST4*A14) - (ST9*A15) + (C8*ST6) - (ST16*ST5) + (C8*
1ST17) - (ST15*ST5) - (C8 *ST7))

D4(5) = ASQ*((-ST9*A14) - (ST13*A15) + (C8*C6*A10) - (ST5*ST6) - (
1ST17*ST5) - (C6*C8*A13))

D4(6) =  -ASQ*((ST13*A14) + (C6*A10*ST5))

D5(1) = ASQ*(((A11 + A12)*C5*ST2) + (ASQ*ST15*ST1) - (2.0*ST15*ST2
1))

D5(2) = ASQ*(((ST6 + ST17 - (2.0*ST7))*ST2) + (ASQ*ST1*ST7))
```

```
D5(3) = ASQ*((-ST4*A16) + ((A10   2.0*A13)*C6*ST2) + (ST18*ST1) + (C8*
1ST15))
D5(4) = ASQ*((-ST4*A17) - (ST9*A16) + ((A11 + (2.0*A12))*C5*C9) +
1(C8*ST7) - (ST15*ST5))
D5(5) = ASQ*((-ST9*A17) - (ST13*A16) + (C9*ST6) + (ST17*C9) + (ST1
18*C8) - (ST5*ST7) + (C9*ST7))
D5(6) = ASQ*((-ST13*A17) + (C6*C9*A10) - (ST18*ST5) + (ST18*C9))
D6(1) = ASQ*((ST16*ST3) + (ST15*ST2))
D6(2) = ASQ*((ST3*ST6) + (ST16*C11 + (ST17*ST3) + (ST15*C11) + (S
1T2*ST7) - (ST7*ST3))
D6(3) = ASQ*((C6*A10*ST3) + (C11*ST6) + (ST17*C11) + (ST18*ST2) -
1(ST18*ST3))
D6(4) = ASQ*((C6*C11*A10) + (ST16*C12)   (ST15*C12) + (ST15*C9) +
1(2.0*ST15*C12))
D6(5) = ASQ*((C12*ST6) + (ST17*C12) + ((C9 + (2.0*C12))*ST7))
D6(6) = ASQ*((C6*C12*A10) + (C6*A13*(C9 + (2.0*C12))))
D7(1) = ASQ*((ST16*C16) + (ST15*C16) + (ST15*ST3))
D7(2) = ASQ*((C16*ST6) + (ST17*C16) + (ST3*ST7) + (ST15*C11))
D7(3) = ASQ*((C6*A10*C16) + (ST18*ST3) + (C11*ST7))
D7(4) = ASQ*((C11*ST18) + (C12*ST15))
D7(5) = ASQ*C12*ST7
D7(6) = ASQ*C12*ST18
D8(1) = ASQ*((C18*ST16) + (C18*ST15) + (C16*ST15) + (C18*ST15))
D8(2) = ASQ*((C18*ST6) + (C18*ST17) + (C16*ST7) + (C18*ST7))
```

```
D8(3) = ASQ*((C6*A10*C18) + (C16*ST18) + (C18*ST18))
D9(1) = ASQ*C18*ST15
D9(2) = ASQ*C18*ST7
D9(3) = ASQ*C18*ST18
WRITE ØUTPUT TAPE 6, 410
WRITE ØUTPUT TAPE 6, 405, D1
WRITE ØUTPUT TAPE 6, 405, D2
WRITE ØUTPUT TAPE 6, 405, D3
WRITE ØUTPUT TAPE 6, 405, D4
WRITE ØUTPUT TAPE 6, 405, D5
WRITE ØUTPUT TAPE 6, 405, D6
WRITE ØUTPUT TAPE 6, 405, D7
WRITE ØUTPUT TAPE 6, 405, D8
WRITE ØUTPUT TAPE 6, 405, D9
ST19 = C1*C5
ST20 = A11 + A12
ST21 = C5*C6*B9 + B10*C5*C5
ST22 = A13*B9 + A12*B10
ST23 = A11*B10 + A10*B9
ST24 = B6 - ASQ*B13
ST25 = C5*C6*B10 + B11*C5*C5
ST26 = 2.0*C5*C6*B7 + C5*C5*B8
ST27 = A13*B10 + A12*B11
ST28 = A11*B11 + A10*B10
```

```
ST29 = B5 - ASQ*B12

ST30 = 2.0*C5*C6*B8 + C6*C6*B7

ST31 = A13*B11 + A10*B11

ST32 = C2 + 2.0*C4

ST33 = A14*C5 + A15*C6

ST34 =  - 2.0*C5

ST35 = C5*C5*A14 + 2.0*C5*C6*A15

ST36 = C6*A14

ST37 = A17*C5 - A16*C6

ST38 = 6.0*(ST2 - ASQ*ST1)

ST39 = C5*C5*A17 + 2.0*C5*C6*A16

ST40 = 2.0*C5*C6*A17 + ST13*A16

ST41 = ST1*ASQ - 4.0*ST2 + 2.0*ST3

ST42 = 2.0*(C9 + C12)

ST43 = ST2 - 2.0*ST3
```

ST44 = C9 + 4.0*C12

ST45 = C16 + 2.0* C18

The reader's attention is directed to the terms ST19 through ST45. These terms are employed to facilitate the solution of equations defining E1(1) through E11(4). It is felt that an example for E1(1), similar to the example for D1(1) stated earlier, would serve no purpose at this point in the reader's understanding of the SOURCE PORGRAM.

The reader may now continue with the SOURCE PROGRAM.

```
E1(1) = ASQ*((2.0*ST4*B9*ST8) - (2.0*ST19*B9*ST20) + (2.0*ST19*A12

1*B9))

E1(2) = ASQ*((ST8 *(2.0*ST21 + ST4*B7)) + (ST12*2.0*ST4*B9) + (2.0*

1C1*(ST15*B10 + B9*ST7)) + (ST4*A12*B2*ASQ) + (C1*C5*(-2.0*(ST22 -S

2T23) - A11*B7 + A12*ST24)) - (2.0*C2*C5*B9*ST20))

E1(3) = ASQ*((ST8 *(2.0*ST25 + ST26)) + (2.0*C1*(ST15*B11 + B10*ST7

1 + ST18*B9)) + (ST12*(2.0*ST21 + ST4*B7)) + (B2*(C5*ST7 + C6*ST15)

2*ASQ) + (C1*(-2.0*C5*(ST27 + ST28))) + (C1*((ST15*ST29) + (ST7*ST2
```

```
34) - (B7 *ST6) - (ST16 *B8))) + (C2 *C5 *(((-2. 0 *ST22 + 2. 0 *ST23)) - A
411 *B7 + A12 *ST24)))
 E1(4) = ASQ *((ST8 *(ST9 *B11 + ST30)) + (ST12 *(2. 0 *ST25 + ST26)) + (
12. 0 *C1 *(B11 *ST7 + ST18 *B10)) + (ASQ *B2 *(ST17 *C5 + C6 *ST7)) + (ASQ *
2B1 *(C5 *ST7 + ST15 *C6)) + (C1 *(-2. 0 *C5 *(ST31 + ST7 *ST29) + ST18 *ST2
34 - B8 *ST6 - C6 *A10 *B7)) + (C2 *(-2. 0 *C5 *(ST27 + ST28) + ST15 *ST29
4+ ST7 *ST24 - B7 *ST6 - C5 *A11 *B8)))
 E1(5) = ASQ *((ST8 *ST13 *B8) + (ST12 *(ST9 *B11 + ST30)) + (ASQ *(B2 *ST
113 *A13 + B1 *(C5 *ST18 + C6 *ST7))) + (ST11 *(A13 *ST29 - A10 *B8)) + (C
22 *(-2. 0 *C5 *(ST31 + ST7 *ST29)) + (ST18 *ST24) - (B8 *ST6) - C6 *A10 *B
37)+ 2. 0 *C1 *ST18 *B11)
 E1(6) = ASQ *(ST12 *ST13 *B8 + B1 *ST13 *A13 *ASQ+ C2 *C6 *(A13 *ST29 - A1
10 *B8))
 E2(1) = ASQ *(2. 0 *ST4 *B9 *A3 - 2. 0 *C5 *B9 *ST20 *C3 - 2. 0 *ST19 *A12 *B9)
 E2(2) = ASQ *(A3 *(2. 0 *ST21 + ST4 *B7) + A4 *2. 0 *ST4 *B9 + ST4 *A12 *B3 *A
1SQ+ ST14 *(-2. 0 *ST22 + 2. 0 *ST23 - A11 *B7 + A12 *ST24) - 2. 0 *C4 *C5 *B
29 *ST20 - 2. 0 *C1 *(ST15 *B10 + B9 *ST7))
 E2(3) = ASQ *(A3 *(2. 0 *ST25 + ST26) + A4 *(2. 0 *ST21 + ST4 *B7) + B3 *(C
15 *ST7 + C6 *ST15) *ASQ+ B4 *ST4 *A12 *ASQ+ C3 *(-2. 0 *C5 *(ST27 + ST28)
2+ ST15 *ST29 + ST7 *ST24 - B7 *ST6 - ST16 *B8) + C4 *C5 *(2. 0 *(-ST22 + S
3T23) - A11 *B7 + A12 *ST24) - 2. 0 *C1 *(ST15 *B11 + B10 *ST7 + ST18 *B9))
 E2(4) = ASQ *(A3 *(ST9 *B11 + ST30) + A4 *(2. 0 *ST25 + ST26) + ASQ *(B3 *
1(C5 *ST18 + C6 *ST7) + B4 *(C5 *ST7 + C6 *ST15)) + C3 *(-2. 0 *C5 *(ST31 +
2 ST7 *ST29) + ST18 *ST24 - B8 *ST6 - C6 *A10 *B7) + C4 *(-2. 0 *C5 *(ST27 +
```

```
 3 ST28) + ST15 *ST29 + ST7 *ST24 - B7 *ST6 - ST16 *B8) - 2. 0 *C1 *(B11 *ST
47 + ST18 *B10))
 E2(5) = ASQ *(A3 *ST13 *B8 + A4 *(ST9 *B11 + ST30) + ASQ *(B3 *ST13 *A13 +
1 B4 *(C5 *ST18 + C6 *ST7)) + C3 *C6 *(A13 *ST29 - A10 *B8) + C4 *(-2. 0 *C5 *
2(ST31 + ST7 *ST29) + ST18 *ST24 - B8 *ST6 - C6 *A10 *B7) - 2. 0 *ST11 *A13
3 *B11)
 E2(6) = ASQ *(A4 *ST13 *B8 + B4 *ST13 *A13 *ASQ + C4 *C6 *(A13 *ST29 - A10 *
1B8) *2. 0)
 E3(1) = ASQ *ST4 *B9 *A5 *2. 0
 E3(2) = ASQ *(2. 0 *A6 *ST4 *B9 + A5 *(2. 0 *ST21 + ST4 *B7) + C1 *ST15 *B9)
 E3(3) = ASQ *(2. 0 *A7 *ST4 *B9 + A6 *(2. 0 *ST21 + ST4 *B7) + A5 *(2. 0 *ST25
1 + ST26) + C1 *(ST15 *B10 + B9 *ST7) + ST32 *ST15 *B9)
 E3(4) = ASQ *(A7 *(2. 0 *ST21 + ST4 *B7) + A6 *(2. 0 *ST25 + ST26) + A5 *(S
1T9 *B11 + ST30) + C1 * (ST15 *B11 + B10 *ST7 + ST1  *B9) + ST32 *(ST15 *B1
20 + B9 *ST7))
 E3(5) = ASQ *(A7 *(2. 0 *ST25 + ST26) + A6 *(ST9 *B11 + ST30) + A5 *ST13 *
1B8 + C1 *(B11 *ST7 + ST18 *B10) + ST32 *(ST15 *B11 + B10 *ST7 + ST18 *B9)
2)
 E3(6) = ASQ *(A7 *(ST9 *B11 + ST30) + A6 *ST13 *B8 + ST11 *A13 *B11 + ST3
12 *(B11 *ST7 + ST18 *B10))
 E3(7) = ASQ *(A7 *ST13 *B8 + ST32 *ST18 *B11)
 E4(1) = ASQ *C3 *ST15 *B9
 E4(2) = ASQ *(C3 *(ST15 *B10 + B9 *ST7) + C4 *ST15 *B9)
 E4(3) = ASQ *(C3 *(ST15 *B11 + B10 *ST7 + ST18 *B9) + C4 *(ST15 *B10 + B9
```

```
1 *ST7))
 E4(4) = ASQ *(C3 *(B11 *ST7 + ST18 *B10) + C4 *(ST15 *B11 + B10 *ST7 + ST
118 *B9))
 E4(5) = ASQ *(C3 *ST18 *B11 + C4 *(B11 *ST7 + ST18 *B10))
 E4(6) = ASQ *C4 *ST18 *B11
 E5(1)   A4H *(2. 0 *C5 *B9 *ST20 *ST1 + 12. 0 *ST1 *ST15 *B9)
 E5(2) = A4H *ST1 *(-C5 *(-2. 0 *(ST22 - ST23) - A11 *B7 + A12 *ST24) + 12
1. 0 *(ST15 *B10 + B9 *ST7))
 E5(3) = ASQ *(-2. 0 *A15 *ST4 *B9 - ST1 *(-2. 0 *C5 *(ST26 + ST28) + ST15 *S
1T29 + ST7 *ST24 - B7 *ST6 - ST16 *B8) - 2. 0 *C8 *C5 *B9 *ST20 + 12. 0 *ASQ *
2ST1 *(ST15 *B11 + B10 *ST7 + ST18 *B9) + 2. 0 *C8 *ST15 *B9)
 E5(4) = ASQ *(-2. 0 *C5 *(B10 *A15 *C5 + B9 *ST33) - ST4 *A12 *B15 *ASQ - ST4
1 *A15 *B7 - (ST1 *(-2. 0 *C5 *ST31 + ST7 *ST29 + ST18 *ST24 - B8 *ST6 - C6 *
2A10 *B7) + C8 *C5 *(-2. 0 *(ST22 - ST23) - A11 *B7 + A12 *ST24) + ST5 *2.
30 *C5 *B9 *ST20) + 12. 0 *ASQ *ST1 *(B11 *ST7 + ST18 *B10) + 2. 0 *C8 *(ST15 *B
410 + B9 *ST7))
 E5(5) = ASQ *(ST34 *(B11 *A15 *C5 + B10 *ST33 + B9 *C6 *A14) - ASQ *(B15 *(
1C5 *ST7 + C6 *ST15) + B14 *ST4 *A12) - B7 *ST35 - B8 *ST4 *A15 - (ST1 *C6 *
2(A13 *ST29 - A10 *B8) + C8 *(ST34 *(ST27 + ST28) + ST15 *ST29 + ST7 *ST2
34 - B7 *ST6 - ST16 *B8) - ST5 *C5 *(-2. 0 *(ST22 - ST23) - A11 *B11 + A12
4 *ST24)) + 12. 0 *ASQ *ST1 *ST18 *B11 + 2. 0 *C8 *(ST15 *B11 + B10 *ST7 + ST1
58 *B9))
 E5(6) = ASQ *(ST34 *(B11 *ST33 + B10 *ST36) - ASQ *(B15 *(C5 *ST18 + C6 *S
1T7) + B14 *(C5 *ST7 + C6 *ST15)) - B7 *(ST9 *A14 + ST13 *A15) - B8 *(ST4 *
```

```
2A14 + ST9 *A15) - (C8 *(ST34 *ST31 + ST7 *ST29 + ST18 *ST24 - B8 *ST6 -
3C6 *A10 *B7) - ST5 *(ST34 *(ST27 + ST28 ) + ST15 *ST29 + ST7 *ST24 - B7 *S
4T6 - ST16 *B8)) + 2. 0 *C8 *(B11 *ST7 + ST18 *B10))
 E5(7) = ASQ *(-ST9 *A14 *B11 - ASQ *(B15 *ST13 *A13 + B14 *(C5 *ST18 + C6 *
1ST7)) - B7 *ST13 *A14 - B8 *(ST9 *A14 + ST13 *A15) - (C6 *C8 *(A13 *ST29 -
2 A10 * B8) - ST5 *(ST34 *ST31 + ST7 *ST29 + ST18 *ST24 - B8 *ST6 - C6 *A10
3 *B7)) + 2. 0 *C8 *ST18 *B11)
 E5(8) = ASQ *(-ASQ *B14 *ST13 *A13 - B8 *ST13 *A14 + C6 *ST5 *(A13 *ST29 -
1A10 *B8))
 E6(1) = ASQ *(ST2 *2. 0 *C5 *B9 *ST20 + 6. 0 *ST15 *B9 *(ST2 - ASQ *ST1))
 E6(2) = ASQ *(-C5 *ST2 *(-2. 0 *(ST22 - ST23) - A11 *B7 + A12 *ST24) + 6.
10 *(ST2 - ASQ *ST1) *(ST15 *B10 + B9 *ST7))
 E6(3) = ASQ *(ST34 *C5 *A16 *B9 - ST2 *(ST34 *(ST27 + ST28 ) + ST15 *ST29
1+ ST7 *ST24 - B7 *ST6 - ST16 *B8) + 6. 0 *(ST2 - ASQ *ST1) *(ST15 *B11 + B
210 *ST7 + ST18 *B9) + ST34 *C8 *A12 *B9)
 E6(4) = ASQ *(ST34 *(B9 * ST37 + B10 *C5 *A16) - B7 *ST4 *A16 - (ST2 *(ST34
1 *ST31 + ST7 *ST29 + ST18 *ST24 - B8 *ST6 - C6 *A10 *B7 + ST34 *C9 *B9 *ST
220) + ST38 *(B11 *ST7 + ST18 *B10) - 2. 0 *C8 *(ST15 *B10 + B9 *ST7))
 E6(5) = ASQ *(ST34 *(B9 *C6 *A17 + B10 *ST37 + B11 *C5*A16) - B7 *ST39 -
1B8 *ST4 *A16 - (C6 *ST2 *(A13 *ST29 - A10 *B8)) + C9 *C5 *(-2. 0 *(ST22 - ST2
23) - A11 *B7 + A12 *ST24) + ST38 *ST18 *B11 - 2. 0 *C8 *(ST15 *B11 + B10 *S
3T7 + ST18 *B9))
 E6(6) = ASQ *(ST34 *(B10 *C6 *A17 + B11 * ST37) - B 7 * ST40 - B8 * ST39 - C9
1 *(ST34 *(ST27 + ST28) + ST15 *ST29 + ST7 *ST24 - B7 *ST6 - ST16 *B8) -
```

```
22. 0*C8*(B11*ST7+ST18*10))
 E6(7) = ASQ*(-ST9*A17*B11 - B7*ST13*A17 - B8*ST40 - C9*(ST34*ST31
1+ ST7*ST29+ST18*ST24 - B8*ST6 - C6*A10*B7) - 2. 0*C6*C8*A13*B11)
 E6(8) = ASQ*(-B8*ST13*A17 - C9*C6*(A13*ST29 - A10*B8))
 E7(1) = ASQ*(ST3*2. 0*C5*B9*ST20+(ST1*ASQ - 4. 0*ST2+2. 0*ST3)*ST
115*B9)
 E7(2) = ASQ*(-ST3*C5*(-2. 0*(ST22 - ST23) - A11*B7+A12*ST24) - C1
11*ST34*B9*ST20+ST41*(ST15*B10+B9*ST7))
 E7(3) = ASQ*(-ST3*(ST34*(ST27+ST28)+ST15*ST29+ST7*ST24 - B7*
1ST6 - ST16*B8) - C11*C5*(-2. 0*(ST22 - ST23) - A11*B7+A12*ST24)+
2ST41*(ST15*B11+B10*ST7+ST18*B9)+C8*ST15*B9)
 E7(4) = ASQ*(-ST3*(ST34*ST31+ST7*ST29+ST18*ST24 - B8*ST6 - C6*
1A10*B7) - C11*(ST34*(ST27+ST28)+ST15*ST29+ST7*ST24 - B7*ST6
2- C5*A11*B8)+2. 0*C5*C12*B9*ST20+ST41*(B11*ST7+ST18*B10)+C8
3*(ST15*B10+B8*ST7)+(ST42 - ST5)*ST15*B9)
 E7(5) = ASQ*(-ST3*C6*(A13*ST29 - A10*B8) - C11*(ST34*ST31+ST7*ST
129 + ST18*ST24 - B8*ST6 - C6*A10*B7) - C5*C12*(-2. 0*(ST22 - ST23)
2- A11*B7+A12*ST24)+ST41*ST18*B11+C8*(ST15*A11+B10*ST7+ST
318*B9)+(ST42 - ST5)*(ST15*B10+B9*ST7))
 E7(6) = ASQ*(-C11*C6*(A13*ST29 - A10*B8) - C12*(ST34*(ST27+ST28)
1 + ST15*ST29+ST7*ST24 - B7*ST6 - ST16*B8)+C8*(B11*ST7+ST18*B
210)+(ST42 - ST5)*(ST15*B11+B10*ST7+ST18*B9))
 E7(7) = ASQ*(-C12*(ST34*ST31+ST7*ST29+ST18*ST24 - B8*ST6 - C6*
1A10*B7)+C8*ST18*B11+(ST42 - ST5)*(B11*ST7+ST18*B10))
```

```
 E7(8) = ASQ*(-C12*C6*(A13*ST29 - A10*B8)+(ST42 - ST5)*ST18*B11)
 E8(1) = ASQ*(2.0*C16*C5*B9*ST20+ST43*ST15*B9)
 E8(2) = ASQ*(-C16*C5*(-2.0*(ST22 - ST23) - A11*B7+A12*ST24)+ST
143*(ST15*B10+B9*ST7))
 E8(3) = ASQ*(-C16*(ST34*(ST27+ST28)+ST15*ST29+ST7*ST24 - B7*
1ST6 - ST16*B8)+ST43*(ST15*B11+B10*ST7+ST18*B9))
 E8(4) = ASQ*(-C16*(ST34*ST31+ST7*ST29+ST18*ST24 - B8*ST6 - C6*
1A10*B7)+ST43*(B11*ST7+ST18*B10)+ST44*ST15*B9)
 E8(5) = ASQ*(-C16*C6*(A13*ST29 - A10*B8)+ST43*ST18*B11+ST44*(S
1T15*B10+B9*ST7))
 E8(6) = ASQ*(ST44*(ST15*B11+B10*ST7+ST18*B9))
 E8(7) = ASQ*ST44*(B11*ST7+ST18*B10)
 E8(8) = ASQ*ST44*ST18*B11
 E9(1) = ASQ*(2.0*C18*C5*B9*ST20+ST3*ST15*B9)
 E9(2) = ASQ*(-C18*C5*(-2.0*(ST22 - ST23) - A11*B7+A12*ST24)+ST
13*(ST15*B10+B9*ST7)+C11*ST15*B9)
 E9(3) = ASQ*(-C18*(ST34*(ST27+ST28)+ST15*ST29+ST7*ST24 - B7*
1ST6 - ST16*B8)+ST3*(ST15*B11+B10*ST7+ST18*B9)+C11*(ST15*B1
20 + B9*ST7))
 E9(4) = ASQ*(-C18*(ST34*ST31+ST7*ST29+ST18*ST24 - B8*ST6 - C6*
1A10*B7)+ST3*(B11*ST7+ST18*B10)+C11*(ST15*B11+B10*ST7+ST1
28*B9)+C12*ST15*B9)
 E9(5) = ASQ*(-C18*(C6*(A13*ST29 - A10*B8))+ST3*ST18*B11+C11*(B
111*ST7+ST18*B10)+C12*(ST15*B10+B9*ST7))
```

```
E9(6) = ASQ*(C11*ST18*B11+C12*(ST15*B11+B10*ST7+ST18*B9))
E9(7) = ASQ*C12*(B11*ST7+ST18*B10)
E9(8) = ASQ*C6*C12*A13*B11
E10(1) = ASQ*ST15*B9*ST45
E10(2) = ASQ*ST45*(ST15*B10+B9*ST7)
E10(3) = ASQ*ST45*(ST15*B11+B10*ST7+ST18*B9)
E10(4) = ASQ*ST45*(B11*ST7+ST18*B10)
E10(5) = ASQ*ST18*B11*ST45
E11(1) = ASQ*ST15*B9*C18
E11(2) = ASQ*C18*(ST15*B10+B9*ST7)
E11(3) = ASQ*C18*(ST15*B11+B10*ST7+ST18*B9)
E11(4) = ASQ*C18*(B11*ST7+ST18*B10)
E11(5) = ASQ*ST18*B11*C18
WRITE ØUTPUT TAPE 6, 413
WRITE ØUTPUT TAPE 6, 405, E1
WRITE ØUTPUT TAPE 6, 405, E2
WRITE ØUTPUT TAPE 6, 405, E3
WRITE ØUTPUT TAPE 6, 405, E4
WRITE ØUTPUT TAPE 6, 405, E5
WRITE ØUTPUT TAPE 6, 405, E6
WRITE ØUTPUT TAPE 6, 405, E7
WRITE ØUTPUT TAPE 6, 405, E8
WRITE ØUTPUT TAPE 6, 405, E9
WRITE ØUTPUT TAPE 6, 405, E10
```

```
      WRITE ØUTPUT TAPE 6, 405, E11
C     DIFFERENTIATE D
      BETA = -3.0
      DØ 2 J = 1, 4
      D1P(J) = D1(J)*BETA
    2 BETA = BETA + 1.0
      BETA = -2.0
      DØ 3 J = 1, 4
      D2P(J) = D2(J)*BETA
    3 BETA = BETA + 1.0
      BETA = -2.0
      DØ 4 J = 1, 5
      D3P(J) = D3(J)*BETA
    4 BETA = BETA + 1.0
      BETA = -5.0
      DØ 5 J = 1, 6
      D4P(J) = D4(J)*BETA
    5 BETA = BETA + 1.0
      BETA = -4.0
      DØ 6 J = 1, 6
      D5P(J) = D5(J)*BETA
    6 BETA = BETA + 1.0
      BETA = -3.0
      DØ 7 J = 1, 6
```

```
      D6P(J) = D6(J)*BETA
    7 BETA = BETA + 1.0
      BETA = -2.0
      DØ8  J = 1, 6
      D7P(J) = D7(J)*BETA
    8 BETA = BETA + 1.0
      BETA = -1.0
      DØ 9 J = 1, 3
      D8P(J) = D8(J)*BETA
    9 BETA = BETA + 1.0
      BETA = 0.0
      DØ 10 J = 1, 3
      D9P(J) = D9(J)*BETA
   10 BETA = BETA + 1.0
C     D1B
      DØ 11 J = 1, 10
   11 D3E1(J) = 0.0
      K = 1
      DØ 12 I = 1, 5
      DØ 13 J = 1, 6
      D3E1(K) = D3E1(K) + (D3(I)*E1(J))
   13 K = K + 1
   12 K = I + 1
      DØ 14 J = 1, 9
```

```
14 E4D1P(J) = 0.0
   K = 1
   DØ 15 I = 1, 6
   DØ 16 J = 1, 4
   E4D1P(K) = E4D1P(K) + (E4(I)*D1P(J))
16 K = K + 1
15 K = I + 1
   D1B(1) = D3E1(1)
   I = 2
   DØ 17 J = 1, 9
   D1B (I) = D3E1(I)-E4D1P(J)
17 I = I + 1
C  D2B
   DØ 18 I = 1, 10
18 D3E2(I) = 0.0
   K = 1
   DØ 19 I = 1, 5
   DØ 20 J = 1, 6
   D3E2(K) = D3E2(K) + (D3(I)*E2(J))
20 K = K + 1
19 K = I + 1
   DØ 21 I = 1, 4
21 D12P(I) = D1(I) + D2P(I)
   DØ 22 I = 1, 9
```

```
   22 E412P(I) = 0.0
      K = 1
      DØ 23 I = 1, 6
      DØ 24 J = 1, 4
      E412P(K) = E412P(K) + (E4(I)*D12P(J))
   24 K = K + 1
   23 K = I + 1
      D2B(1) = D3E2(1)
      I = 2
      DØ 25 J = 1, 9
      D2B(I) = D3E2(I) - E412P(J)
   25 I = I + 1
C     D3B
      DØ 26 I = 1, 11
   26 D3E3(I) = 0.0
      K = 1
      DØ 27 I = 1, 5
      DØ 28 J = 1, 7
      D3E3(K) = D3E3(K) + (D3(I)*E3(J))
   28 K = K + 1
   27 K = I + 1
      D23P(1) = D3P(1)
      I = 2
      DØ 29 J = 1, 4
      D23P(I) = D2(J) + D3P(I)
```

```
29 I = I + 1
   DØ 30 I = 1, 10
30 E423P(I) = 0.0
   K = 1
   DØ 31 I = 1, 6
   DØ 32 J = 1, 5
   E423P(K) = E423P(K) + (E4(I)*D23P(J))
32 K = K + 1
31 K = I + 1
   D3B(1) = D3E3(1)
   I = 2
   DØ 33 J = 1, 10
   D3B(I) = D3E3(I) - E423P(J)
33 I = I + 1
C  E1B
   DØ 34 I = 1, 12
34 D3E5(I) = 0.0
   K = 1
   DØ 35 I = 1, 5
   DØ 36 J = 1, 8
   D3E5(K) = D3E5(K) + (D3(I)*E5(J))
36 K = K + 1
35 K = I + 1
   DØ 37 I = 1, 11
```

```
37  E4D4P(I) = 0.0
    K = 1
    DØ 38 I = 1, 6
    DØ 39 J = 1, 6
    E4D4P(K) = E4D4P(K) + (E4(I) *D4P(J))
39  K = K + 1
38  K = I + 1
    E1B(1) = D3E5(1)
    I = 2
    DØ 40 J = 1, 11
    E1B(I) = D3E5(I) - E4D4P(J)
40  I = I + 1
C   E2B
    DØ 41 I = 1, 12
41  D3E6(I) = 0.0
    K = 1
    DØ 42 I = 1, 5
    DØ 43 J = 1, 8
    D3E6(K) = D3E6(K) + (D3(I) *E6(J))
43  K = K + 1
42  K = I + 1
    DØ 44 I = 1, 6
44  D45P(I) = D4(I) + D5P(I)
    DØ 45 I = 1, 11
```

```
45 E445P(I) = 0. 0
   K = 1
   DØ 46 I = 1, 6
   DØ 47 J = 1, 6
   E445P(K) = E445P(K) + (E4(I)*D45P(J))
47 K = K + 1
46 K = I + 1
   E2B(1) = D3E6(1)
   I = 2
   DØ 48 J = 1, 11
   E2B(I) = D3E6(I) - E445P(J)
48 I = I + 1
C  E3B
   DØ 49 I = 1, 12
49 D3E7(I) = 0. 0
   K = 1
   DØ 50 I = 1, 5
   DØ 51 J = 1, 8
   D3E7(K) = D3E7(K)+ (D3(I)*E7(J))
51 K = K + 1
50 K = I + 1
   DØ 52 I = 1, 6
52 D56P(I) = D5(I) + D6P(I)
   DØ 53 I = 1, 11
```

```
   53 E456P(I) = 0.0
      K = 1
      DØ 54 I = 1, 6
      DØ 55 J = 1, 6
      E456P(K) = E456P(K) + (E4(I)*D56P(J))
   55 K = K + 1
   54 K = I + 1
      E3B(1) = D3E7(1)
      I = 2
      DØ 56 J = 1, 11
      E3B(I) = D3E7(I) - E456P(J)
   56 I = I + 1
C     E4B
      DØ 57 I = 1, 12
   57 D3E8(I) = 0.0
      K = 1
      DØ 58 I = 1, 5
      DØ 59 J = 1, 8
      D3E8(K) = D3E8(K) + (D3(I)*E8(J))
   59 K = K + 1
   58 K = I + 1
      DØ 60 I = 1, 6
   60 D67P(I) = D6(I) + D7P(I)
      DØ 61 I = 1, 11
```

```
61 E467P(I) = 0.0
   K = 1
   DØ 62 I = 1, 6
   DØ 63 J = 1, 6
   E467P(K) = E467P(K) + (E4(I)*D67P(J))
63 K = K + 1
62 K = I + 1
   E4B(1) = D3E8(1)
   I = 2
   DØ 64 J = 1, 11
   E4B(I) = D3E8(I) - E467P(J)
64 I = I + 1
C  E5B
   DØ 65 I = 1, 12
65 D3E9(I) = 0.0
   K = 1
   DØ 66 I = 1, 5
   DØ 67 J = 1, 8
   D3E9(K) = D3E9(K) + (D3(I)*E9(J))
67 K = K + 1
66 K = I + 1
   DØ 68 I = 1, 3
68 D78P(I) = D7(I) + D8P(I)
   DØ 69 I = 4, 6
69 D78P(I) = D7(I)
   DØ 70 I = 1, 11
```

```
70 E478P(I) = 0.0
   K = 1
   DØ 71 I = 1, 6
   DØ 72 J = 1, 6
   E478P(K) = E478P(K) + (E4(I)*D78P(J))
72 K = K + 1
71 K = I + 1
   E5B(1) = D3E9(1)
   I = 2
   DØ 73 J = 1, 11
   E5B(I) = D3E9(I) - E478P(J)
73 I = I + 1
C  E6B
   DØ 74 I = 1, 9
74 D3E10(I) = 0.0
   K = 1
   DØ 75 I = 1, 5
   DØ 76 J = 1, 5
   D3E10(K) = D3E10(K) + (D3(I)*E10(J))
76 K = K + 1
75 K = I + 1
   DØ 77 I = 1, 3
77 D89P(I) = D8(I) + D9P(I)
   DØ 78 I = 1, 8
```

```
78  E489P(I) = 0.0
    K = 1
    DØ 79 I = 1, 6
    DØ8 0 J = 1, 3
    E489P(K) = E489P(K) + (E4(I) *D89P(J))
80  K = K + 1
79  K = I + 1
    E6B(1) = D3E10(1)
    I = 2
    DØ 81 J = 1, 8
    E6B(I) = D3E10(I) + E489P(J)
81  I = I + 1
C   E7B
    DØ 82 I = 1, 9
82  D3E11(I) = 0.0
    K = 1
    DØ 83 I = 1, 5
    DØ 84 J = 1, 5
    D3E11(K) = D3E11(K) + (D3(I) *E11(J))
84  K = K + 1
83  K = I + 1
    DØ 85 I = 1, 8
85  E4D9(I) = 0.0
    K = 1
```

```
      DØ 86 I = 1, 6
      DØ 87 J = 1, 3
      E4D9(K) = E4D9(K) + (E4(I)*D9(J))
   87 K = K + 1
   86 K = I + 1
      E7B(1) = D3E11(1)
      I = 2
      DØ 88 J = 1, 7
      E7B(I) = D3E11(I) - E4D9(J)
   88 I = I + 1
      WRITE ØUTPUT TAPE 6, 414
      WRITE ØUTPUT TAPE 6, 405, D1B
      WRITE ØUTPUT TAPE 6, 405, D2B
      WRITE ØUTPUT TAPE 6, 405, D3B
      WRITE ØUTPUT TAPE 6, 415
      WRITE ØUTPUT TAPE 6, 405, E1B
      WRITE ØUTPUT TAPE 6, 405, E2B
      WRITE ØUTPUT TAPE 6, 405, E3B
      WRITE ØUTPUT TAPE 6, 405, E4B
      WRITE ØUTPUT TAPE 6, 405, E5B
      WRITE ØUTPUT TAPE 6, 405, E6B
      WRITE ØUTPUT TAPE 6, 405, E7B
C     F1B
      DØ 89 I = 1, 16
   89 D3BD4(I) = 0.0
```

```
      K = 1
      DØ 90 I = 1, 11
      DØ 91 J = 1, 6
      D3BD4(K) = D3BD4(K) + (D3B(I)*D4(J))
   91 K = K + 1
   90 K = I + 1
      DØ 92 I = 1, 16
   92 D3E1B(I) = 0.0
      K = 1
      DØ 93 I = 1, 5
      DØ 94 J = 1, 12
      D3E1B(K) = D3E1B(K) + (D3(I)*E1B(J))
   94 K = K + 1
   93 K + I + 1
      DØ 95 I = 1, 16
   95 F1B(I) = D3BD4(I) - D3E1B(I)
C     F2B
      DØ 96 I = 1, 16
   96 D3BD5(I) = 0.0
      K = 1
      DØ 97 I = 1, 11
      DØ 98 J = 1, 6
      D3BD5(K) = D3BD5(K) + (D3B(I)*D5(J))
   98 K = K + 1
```

```
   97 K = I + 1
      DØ 99 I = 1, 16
   99 D3E2B(I) = 0.0
      K = 1
      DØ 100 I = 1, 5
      DØ 101 J + 1, 12
      D3E2B(K) = D3E2B(K) + (D3(I)*E2B(J))
  101 K = K + 1
  100 K = I + 1
      DØ 102 I = 1, 16
  102 F2B(I) = D3BD5(I) - D3E2B(I)
C     F3B
      DØ 103 I = 1, 16
  103 D3BD6(I) = 0.0
      K = 1
      DØ 104 I = 1, 11
      DØ 105 J = 1, 6
      D3BD6(K) = D3BD6(K) + (D3B(I)*D6(J))
  105 K = K + 1
  104 K = I + 1
      DØ 106 I = 1, 16
  106 D3E3B(I) = 0.0
      K = 1
      DØ 107 I = 1, 5
```

```
      DØ 108 J = 1, 12
      D3E3B(K) = D3E3B(K) + (D3(I)*E3B(J))
  108 K = K + 1
  107 K = I + 1
      DØ 109 I = 1, 16
  109 F3B(I) = D3BD6(I) - D3E3B(I)
C     F4B
      DØ 110 I = 1, 16
  110 D3BD7(I) = 0.0
      K = 1
      DØ 111 I = 1, 11
      DØ 112 J = 1, 6
      D3BD7(K) = D3BD7(K) + (D3B(I)*D7(J))
  112 K = K + 1
  111 K = I + 1
      DØ 113 I = 1, 16
  113 D3E4B(I) = 0.0
      K = 1
      DØ 114 I = 1, 5
      DØ 115 J = 1, 12
      D3E4B(K) = D3E4B(K) + (D3(I)*E4B(J))
  115 K = K + 1
  114 K = I + 1
      DØ 116 I = 1, 16
```

```
116 F4B(I) = D3BD7(I) - D3E4B(I)
C       F5B
        DØ 117 I = 1, 13
117 D3BD8(I) = 0.0
        K = 1
        DØ 118 I = 1, 11
        DØ 119 J = 1, 3
        D3BD8(K) = D3BD8(K) + (D3B(I)*D8(J))
119 K = K + 1
118 K = I + 1
        DØ 120 I = 1, 16
120 D3E5B(I) = 0.0
        K = 1
        DØ 121 I = 1, 5
        DØ 122 J = 1, 12
        D3E5B(K) = D3E5B(K) + (D3(I)*E5B(J))
122 K = K + 1
121 K = I + 1
        DØ 123 I = 1, 13
123 F5B(I) = D3BD8(I) - D3E5B(I)
        DØ 124 I = 14, 16
124 F5B(I) = -D3E5B(I)
C       F6B
        DØ 125 I = 1, 13
```

```
  125 D3BD9(I) = 0.0
      K = 1
      DØ 126 I = 1, 11
      DØ 127 J = 1, 3
      D3BD9(K) = D3BD9(K) + (D3B(I)*D9(J))
  127 K = K + 1
  126 K = I + 1
      DØ 128 I = 1, 13
  128 D3E6B(I) = 0.0
      K = 1
      DØ 129 I = 1, 5
      DØ 130 J = 1, 9
      D3E6B(K) = D3E6B(K) + (D3(I)*E6B(J))
  130 K = K + 1
  129 K = I + 1
      DØ 131 I = 1, 13
  131 F6B(I) = D3BD9(I) - D3E6B(I)
C     F7B
      DØ 132 I = 1, 13
  132 D3E7B(I) = 0.0
      K = 1
      DØ 133 I = 1, 5
      DØ 134 J = 1, 9
      D3E7B(K) = D3E7B(K) + (D3(I)*E7B(J))
```

```
134 K = K + 1
133 K = I + 1
    DØ 135 I = 1, 13
135 F7B(I) = -D3E7B(I)
C       F1BP
    BETA = -10.0
    DØ 136 I = 1, 16
    F1BP(I) = BETA*F1B(I)
136 BETA = BETA + 1.0
C       F2BP
    BETA = -9.0
    DØ 137 I = 1, 16
    F2BP(I) = F2B(I)*BETA
137 BETA = BETA + 1.0
C       F3BP
    BETA =  3.0
    DØ 138 I = 1, 16
    F3BP(I) = F3B(I)*BETA
138 BETA = BETA + 1.0
C       F4BP
    BETA = -7.0
    DØ 139 I = 1, 16
    F4BP(I) = F4B(I)*BETA
139 BETA = BETA + 1.0
```

```
C       F5BP
        BETA = -6.0
        DØ 140 I = 1, 16
        F5BP(I) = F5B(I)*BETA
  140   BETA = BETA + 1.0
C       F6BP
        BETA = -5.0
        DØ 141 I = 1, 13
        F6BP(I) = F6B(I)*BETA
  141   BETA = BETA + 1.0
C       F7BP
        BETA = -4.0
        DØ 142 I = 1, 13
        F7BP(I) = F7B(I)*BETA
  142   BETA = BETA + 1.0
C       R1
        DØ 143 I = 1, 14
  143   D1D3B(I) = 0.0
        K = 1
        DØ 144 I = 1, 4
        DØ 145 J = 1, 11
        D1D3B(K) = D1D3B(K) + (D1(I)*D3B(J))
  145   K = K + 1
  144   K = I + 1
```

```
      DØ 146 I = 1, 14
  146 D1BD3(I) = 0.0
      K = 1
      DØ 147 I = 1, 10
      DØ 148 J = 1, 5
      D1BD3(K) = D1BD3(K) + (D1B(I)*D3(J))
  148 K = K + 1
  147 K + I + 1
      DØ 149 I = 1, 14
  149 R1(I) = D1D3B(I) - D1BD3(I)
C     R2
      DØ 150 I = 1, 14
  150 D2D3B(I) = 0.0
      K = 1
      DØ 151 I = 1, 4
      DØ 152 J = 1, 11
      D2D3B(K) = D2D3B(K) + (D2(I)*D3B(J))
  152 K = K + 1
  151 K = I + 1
      DØ 153 I = 1, 14
  153 D2BD3(I) = 0.0
      K = 1
      DØ 154 I = 1, 10
      DØ 155 J = 1, 5
```

```
      D2BD3(K) = D2BD3(K) + (D2B(I)*D3(J))
  155 K = K + 1
  154 K = I + 1
      DØ 156 I = 1, 14
  156 R2(I) = D2D3B(I) - D2BD3(I)
C     R1P
      BETA = -8.0
      DØ 157 I = 1, 14
      R1P(I) = R1(I)*BETA
  157 BETA = BETA + 1.0
C     R2P
      BETA = -7.0
      DØ 158 I = 1, 14
      R2P(I) = R2(I)*BETA
  158 BETA = BETA + 1.0
C     Y1
      DØ 159 I = 1, 17
  159 R2D1(I) = 0.0
      K = 1
      DØ 160 I = 1, 14
      DØ 161 J = 1, 4
      R2D1(K) = R2D1(K) + (R2(I)*D1(J))
  161 K = K + 1
  160 K = I + 1
```

```
      DØ 162 I = 1, 17
  162 R1PD2(I) = 0.0
      K = 1
      DØ 163 I = 1, 14
      DØ 164 J = 1, 4
      R1PD2(K) = R1PD2(K) + (R1P(I)*D2(J))
  164 K = K + 1
  163 K = I + 1
      Y1(1) = -R1PD2(1)
      I = 2
      DØ 165 J = 1, 16
      Y1(I) = R2D1(J) - R1PD2(I)
  165 I = I + 1
      Y1(18) = R2D1(17)
C     Y2
      DØ 166 I = 1, 17
  166 R2D2(I) = 0.0
      K = 1
      DØ 167 I = 1, 14
      DØ 168 J = 1, 4
      R2D2(K) = R2D2(K) + (R2(I)*D2(J))
  168 K = K + 1
  167 K = I + 1
      DØ 169 I = 1, 14
```

```
169 R12(I) = R1(I) + R2P(I)
    DØ 170 I = 1, 18
170 D312K(I) = 0.0
    K = 1
    DØ 171 I = 1, 5
    DØ 172 J = 1, 14
    D312K(K) = D312K(K) + (D3(I)*R12(J))
172 K = K + 1
171 K = I + 1
    Y2(1) = -D312K(1)
    I = 2
    DØ 173 J = 1, 17
    Y2(I) = R2D2(J) - D312K(I)
173 I = I + 1
C   Y3
    DØ 174 I = 1, 31
174 R1Y2(I) = 0.0
    K = 1
    DØ 175 I = 1, 14
    DØ 176 J = 1, 18
    R1Y2(K) = R1Y2(K) + (R1(I)*Y2(J))
176 K = K + 1
175 K = I + 1
    DØ 177 I = 1, 31
```

```
  177 R2Y1(I) = 0.0
      K = 1
      DØ 178 I = 1, 14
      DØ 179 J = 1, 18
      R2Y1(K) = R2Y1(K) + (R2(I)*Y1(J))
  179 K = K + 1
  178 K = I + 1
      DØ 180 I = 1, 31
  180 Y3(I) = R1Y2(I) - R2Y1(I)
C     Y1P
      BETA = - 11.0
      DØ 181 I = 1, 18
      Y1P(I) = Y1(I)*BETA
  181 BETA = BETA + 1.0
C     Y2P
      BETA = -10.0
      DØ 182 I = 1, 18
      Y2P(I) = Y2(I)*BETA
  182 BETA = BETA + 1.0
C     Y3P
      BETA = -18.0
      DØ 183 I = 1, 31
      Y3P(I) = Y3(I)*BETA
  183 BETA = BETA + 1.0
```

```
      WRITE ØUTPUT TAPE 6, 416
      WRITE ØUTPUT TAPE 6, 405, F1B
      WRITE ØUTPUT TAPE 6, 405, F2B
      WRITE ØUTPUT TAPE 6, 405, F3B
      WRITE ØUTPUT TAPE 6, 405, F4B
      WRITE ØUTPUT TAPE 6, 405, F5B
      WRITE ØUTPUT TAPE 6, 405, F6B
      WRITE ØUTPUT TAPE 6, 405, F7B
      WRITE ØUTPUT TAPE 6, 417
      WRITE ØUTPUT TAPE 6, 405, R1
      WRITE ØUTPUT TAPE 6, 405, R2
      WRITE ØUTPUT TAPE 6, 418
      WRITE ØUTPUT TAPE 6, 405, Y1
      WRITE ØUTPUT TAPE 6, 405, Y2
      WRITE ØUTPUT TAPE 6, 405, Y3
C     R1B
      DØ 184 I = 1, 48
  184 Y2PY3(I) = 0.0
      K = 1
      DØ 35 I = 1, 18
      DØ 186 J = 1, 31
      Y2PY3(K) = Y2PY3(K) + (Y2P(I)*Y3(J))
  186 K = K + 1
  185 K = I + 1
```

```
      DØ 187 I = 1, 48
  187 Y2Y3P(I) = 0.0
      K = 1
      DØ 188 I = 1, 18
      DØ 189 J = 1, 31
      Y2Y3P(K) = Y2Y3P(K) + (Y2(I)*Y3P(J))
  189 K = K + 1
  188 K = I + 1
      DØ 190 I = 1, 48
  190 Y1Y3(I) = 0.0
      K = 1
      DØ 191 I = 1, 18
      DØ 192 J = 1, 31
      Y1Y3(K) = Y1Y3(K) + (Y1(I)*Y3(J))
  192 K = K + 1
  191 K = I + 1
      DØ 193 I = 1, 48
  193 R1B(I) = Y2PY3(I) - Y2Y3P(I) + Y1Y3(I)
C     R2B
      DØ 194 I = 1, 48
  194 R2B(I) = 0.0
      K = 1
      DØ 195 I = 1, 18
      DØ 196 J = 1, 31
```

```
      R2B(K) = R2B(K) + (Y2(I)*Y3(J))
  196 K = K + 1
  195 K = I + 1
C     R3B
      DØ 197 I = 1, 44
  197 R2PY3(I) = 0.0
      K = 1
      DØ 198 I = 1, 14
      DØ 199 J = 1, 31
      R2PY3(K) = R2PY3(K) + (R2P(I)*Y3(J))
  199 K = K + 1
  198 K = I + 1
      DØ 200 I = 1, 44
  200 R2Y3P(I) = 0.0
      K = 1
      DØ 201 I = 1, 14
      DØ 202 J = 1, 31
      R2Y3P(K) = R2Y3P(K) + (R2(I) *Y3(J))
  202 K = K + 1
  201 K = I + 1
      DØ 203 I = 1, 44
  203 R1Y3(I) = 0.0
      K = 1
      DØ 204 I = 1, 14
```

```
      DØ 205 J = 1, 31
      R1Y3(K) = R1Y3(K) + (R1(I)*Y3(J))
  205 K = K + 1
  204 K = I + 1
      DØ 206 I = 1, 44
  206 R3B(I) = -R2PY3(I) + R2Y3P(I) - R1Y3(I)
C     R4B
      DØ 207 I = 1, 44
  207 R4B(I) = 0.0
      K = 1
      DØ 208 I = 1, 14
      DØ 209 J = 1, 31
      R4B(K) = R4B(K) - (R2(I)*Y3(J))
  209 K = K + 1
  208 K = I + 1
C     G1B
      DØ 210 I = 1, 19
  210 R2D4(I) = 0.0
      K = 1
      DØ 211 I = 1, 14
      DØ 212 J = 1, 6
      R2D4(K) = R2D4(K) + (R2(I)*D4(J))
  212 K = K + 1
  211 K = I + 1
```

```
      DØ 213 I = 1, 20
  213 D3F1P(I) = 0.0
      K = 1
      DØ 214 I = 1, 5
      DØ 215 J = 1, 16
      D3F1P(K) = D3F1P(K) + (D3(I)*F1BP(J))
  215 K = K + 1
  214 K = I + 1
      G1B(1) = -D3F1P(1)
      I = 2
      DØ 216 J = 1, 19
       G1B(I) = R2D4(J) - D3F1P(I)
  216 I = I + 1
C     G2B
      DØ 217 I = 1, 19
  217 R2D5(I) = 0.0
      K = 1
      DØ 218 I = 1, 14
      DØ 219 J = 1, 6
      R2D5(K) = R2D5(K) + (R2(I)*D5(J))
  219 K = K + 1
  218 K = I + 1
      DØ 220 I = 1, 16
```

```
220 F12(I) = F1B(I) + F2BP(I)
    DØ 221 I = 1, 20
221 D312(I) = 0.0
    K = 1
    DØ 222 I = 1, 5
    DØ 223 J = 1, 16
    D312(K) = D312(K) + (D3(I)*F12(J))
223 K = K + 1
222 K = I + 1
    G2B(1) = -D312(1)
    I = 2
    DØ 224 J = 1, 19
     G2B(I) = R2D5(J) - D312(I)
224 I = I + 1
C       G3B
    DØ 225 I = 1, 19
225 R2D6(I) = 0.0
    K = 1
    DØ 226 I = 1, 14
    DØ 227 J = 1, 6
    R2D6(K) = R2D6(K) + (R2(I)*D6(J))
227 K = K + 1
226 K = I + 1
    DØ 228 I = 1, 16
```

```
228 F23(I) = F2B(I) + F3BP(I)
    DØ 229 I = 1, 20
229 D323(I) = 0.0
    K = 1
    DØ 230 I = 1, 5
    DØ 231 J = 1, 16
    D323(K) = D323(K) + (D3(I)*F23(J))
231 K = K + 1
230 K = I + 1
    G3B(1) = -D323(1)
    I = 2
    DØ 232 J = 1, 19
    G3B(I) = R2D6(J) - D323(I)
232 I = I + 1
C   G4B
    DØ 233 I = 1, 19
233 R2D7(I) = 0.0
    K = 1
    DØ 234 I = 1, 14
    DØ 235 J = 1, 6
    R2D7(K) = R2D7(K) + (R2(I)*D7(J))
235 K = K + 1
234 K = I + 1
    DØ 236 I = 1, 16
```

```
236 F34(I) = F3B(I) + F4BP(I)
    DØ 237 I = 1, 20
237 D334(I) = 0.0
    K = 1
    DØ 238 I = 1, 5
    DØ 239 J = 1, 16
    D334(K) = D334(K) + (D3(I)*F34(J))
239 K = K + 1
238 K = I + 1
    G4B(1) = - D334(1)
    I = 2
    DØ 240 J = 1, 19
    G4B(I) = R2D7(J) - D334(I)
240 I = I + 1
C   G5B
    DØ 241 I = 1, 16
241 R2D8(I) = 0.0
    K = 1
    DØ 242 I = 1, 14
    DØ 243 J = 1, 3
    R2D8(K) = R2D8(K) + (R2(I)*D3(J))
243 K = K + 1
242 K = I + 1
    DØ 244 I = 1, 16
```

```
244 F45(I) = F4B(I) + F5BP(I)
    DØ 245 I = 1, 20
245 D345(I) = 0.0
    K = 1
    DØ 246 I = 1, 5
    DØ 247 J = 1, 16
    D345(K) = D345(K) + (D3(I)*F45(J))
247 K = K + 1
246 K = I + 1
    G5B(1) = -D345(1)
    I = 2
    DØ 248 J = 1, 16
    G5B(I) = R2D8(J) - D345(I)
248 I = I + 1
    DØ 249 I = 17, 20
249 G5B(I) = -D345(I)
C   G6B
    DØ 250 I = 1, 16
250 R2D9(I) = 0.0
    K = 1
    DØ 251 I = 1, 14
    DØ 252 J = 1, 3
    R2D9(K) = R2D9(K) + (R2(I)*D9(J))
252 K = K + 1
```

```
251 K = I + 1
    DØ 253 I = 1, 13
253 F56(I) = F5B(I) + F6BP(I)
    DØ 254 I = 14, 16
254 F56(I) = F5B(I)
    DØ 255 I = 1, 20
255 D356(I) = 0.0
    K = 1
    DØ 256 I = 1, 5
    DØ 257 J = 1, 16
    D356(K) = D356(K) + (D3(I)*F56(J))
257 K = K + 1
256 K = I + 1
    G6B(1) = -D356(1)
    I = 2
    DØ 258 J = 1, 16
    G6B(I) = R2D9(J) - D356(I)
258 I = I + 1
    DØ 259 I = 18, 20
259 G6B(I) = -D356(I)
C       G7B
    DØ 260 I = 1, 13
260 F67(I) = F6B(I) + F7BP(I)
    DØ 261 I = 1, 17
```

```
  261 G7B(I) = 0.0
      K = 1
      DØ 262 I = 1, 5
      DØ 263 J = 1, 13
      G7B(K) = G7B(K) - (D3(I)*F67(J))
  263 K = K + 1
  262 K = I + 1
C     G8B
      DØ 264 I = 1, 17
  264 G8B(I) = 0.0
      K = 1
      DØ 265 I = 1, 5
      DØ 266 J = 1, 13
      G8B(K) = G8B(K) - (D3(I)*F7B(J))
  266 K = K + 1
  265 K = I + 1
C     G1BP
      BETA = -13.0
      DØ 267 I = 1, 20
      G1BP(I) = G1B(I)*BETA
  267 BETA = BETA + 1.0
C     G2BP
      BETA = -12.0
      DØ 268 I = 1, 20
```

```
      G2BP(I) = G2B(I)*BETA
  268 BETA = BETA + 1.0
C     G3BP
      BETA = -11.0
      DØ 269 I = 1, 20
      G3BP(I) = G3B(I)*BETA
  269 BETA = BETA + 1.0
C     G4BP
      BETA = -10.0
      DØ 270 I = 1, 20
      G4BP(I) = G4B(I)*BETA
  270 BETA = BETA + 1.0
C     G5BP
      BETA = -9.0
      DØ 271 I = 1, 20
      G5BP(I) = G5B(I)*BETA
  271 BETA = BETA + 1.0
C     G6BP
      BETA = -8.0
      DØ 272 I = 1, 20
      G6BP(I) = G6B(I)*BETA
  272 BETA = BETA + 1.0
C     G7BP
      BETA = -7.0
```

```
      DØ 273 I = 1, 17
      G7BP(I) = G7B(I)*BETA
  273 BETA = BETA + 1.0
C     G8BP
      BETA = -6.0
      DØ 274 I = 1, 17
      G8BP(I) = G8B(I)*BETA
  274 BETA = BETA + 1.0
      WRITE ØUTPUT TAPE 6, 420
      WRITE ØUTPUT TAPE 6, 405, R1B
      WRITE ØUTPUT TAPE 6, 405, R2B
      WRITE ØUTPUT TAPE 6, 405, R3B
      WRITE ØUTPUT TAPE 6, 405, R4B
      WRITE ØUTPUT TAPE 6, 423
      WRITE ØUTPUT TAPE 6, 405, G1B
      WRITE ØUTPUT TAPE 6, 405, G2B
      WRITE ØUTPUT TAPE 6, 405, G3B
      WRITE ØUTPUT TAPE 6, 405, G4B
      WRITE ØUTPUT TAPE 6, 405, G5B
      WRITE ØUTPUT TAPE 6, 405, G6B
      WRITE ØUTPUT TAPE 6, 405, G7B
      WRITE ØUTPUT TAPE 6, 405, G8B
C     T1
      DØ 275 I = 1, 60
```

```
275 T1(I) = 0.0
    K = 1
    DØ 276 I = 1, 44
    DØ 277 J = 1, 17
    T1(K) = T1(K) + (R4B(I)*G8B(J))
277 K = K + 1
276 K = I + 1
    WRITE ØUTPUT TAPE 6, 424
    WRITE ØUTPUT TAPE 6, 405, T1
C   T2
    DØ 278 I = 1, 60
278 R2F7(I) = 0.0
    K = 1
    DØ 279 I = 1, 48
    DØ 280 J = 1, 13
    R2F7(K) = R2F7(K) + (R2B(I)*F7B(J))
280 K = K + 1
279 K = I + 1
    DØ 281 I = 1, 60
281 R3G8(I) = 0.0
    K = 1
    DØ 282 I = 1, 44
    DØ 283 J = 1, 17
    R3G8(K) = R3G8(K) + (R3B(I)*G8B(J))
283 K = K + 1
```

```
282 K = I + 1
    DØ 284 I = 1, 17
284 G78(I) = G7B(I) + G8BP(I)
    DØ 285 I = 1, 60
285 R478(I) = 0.0
    K = 1
    DØ 286 I = 1,44
    DØ 287 J = 1, 17
    R478(K) = R478(K) + (R4B(I)*G78(J))
287 K = K + 1
286 K = I + 1
    DØ 288 I = 1, 60
288 T2(I) = R2F7(I) + R3G8(I) + R478(I)
    WRITE ØUTPUT TAPE 6, 426
    WRITE ØUTPUT TAPE 6, 405, T2
C   T3
    DØ 289 I = 1, 60
289 R1F7(I) = 0.0
    K = 1
    DØ 290 I = 1, 48
    DØ 291 J = 1, 13
    R1F7(K) = R1F7(K) + (R1B(I)*F7B(J))
291 K + K + 1
290 K = I + 1
```

```
      DØ 292 I = 1, 60
  292 R267Z(I) = 0.0
      K = 1
      DØ 293 I = 1, 48
      DØ 294 J = 1, 13
      R267Z(K) = R267Z(K) + (R2B(I)*F67(J))
  294 K = K + 1
  293 K = I + 1
      DØ 295 I = 1, 60
  295 R3G7(I) = 0.0
      K = 1
      DØ 296 I = 1, 44
      DØ 297 J = 1, 17
      R3G7(K) = R3G7(K) + (R3B(I)*G7B(J))
  297 K = K + 1
  296 K = I + 1
      DØ 298 I = 1, 17
  298 G67(I) = G6B(I) + G7BP(I)
      DØ 299 I = 18, 20
  299 G67(I) = G6B(I)
      DØ 300 I = 1, 63
  300 R467G(I) = 0.0
      K = 1
      DØ 301 I = 1, 44
      DØ 302 J = 1, 20
      R467G(K) = R467G(K) + (R4B(I)*G67(J))
```

```
302 K = K + 1
301 K = I + 1
    DØ 303 I = 1, 60
303 T3(I) = R1F7(I) + R267Z(I) + R3G7(I) + R467G(I)
    DØ 304 I = 61, 63
304 T3(I) = R467G(I)
    WRITE ØUTPUT TAPE 6, 427
    WRITE ØUTPUT TAPE 6, 405, T3
C   T4
    DØ 305 I = 1, 60
305 R1F6(I) = 0.0
    K = 1
    DØ 306 I = 1, 48
    DØ 307 J = 1, 13
    R1F6(K) = R1F6(K) + (R1B(I)*F6B(J))
307 K = K + 1
306 K = I + 1
    DØ 308 I = 1, 63
308 R256Z(I) = 0.0
    K = 1
    DØ 309 I = 1, 48
    DØ 310 J = 1, 16
    R256Z(K) = R256Z(K) + (R2B(I)*F56(J))
310 K = K + 1
```

```
309 K = I + 1
    DØ 311 I = 1, 63
311 R3G6(I) = 0.0
    K = 1
    DØ 312 I = 1, 44
    DØ 313 J = 1, 20
    R3G6(K) = R3G6(K) + (R3B(I)*G6B(J))
313 K = K + 1
312 K = I + 1
    DØ 314 I = 1, 20
314 G56(I) = G5B(I) + G6BP(I)
    DØ 315 I = 1, 63
315 R456G(I) = 0.0
    K = 1
    DØ 316 I = 1, 44
    DØ 317 J = 1, 20
    R456G(K) = R456G(K) + (R4B(I)*G56(J))
317 K = K + 1
316 K = I + 1
    DØ 318 I = 1, 60
318 T4(I) = R1F6(I) + R256Z(I) + R3G6(I) + R456G(I)
    DØ 319 I = 61, 63
319 T4(1) = R256Z(I) + R3G6(I) + R456G(I)
    WRITE ØUTPUT TAPE 6, 429
```

```
      WRITE ØUTPUT TAPE 6, 405, T4
C     T5
      DØ 320 I = 1, 63
  320 R1F5(I) = 0.0
      K = 1
      DØ 321 I = 1, 48
      DØ 322 J = 1, 16
      R1F5(K) = R1F5(K) + (R1B(I)*F5B(J))
  322 K = K + 1
  321 K = I + 1
      DØ 323 I = 1, 63
  323 R245Z(I) = 0.0
      K = 1
      DØ 324 I = 1, 48
      DØ 325 J = 1, 16
      R245Z(K) = R245Z(K) + (R2B(I)*F45(J))
  325 K = K + 1
  324 K = I + 1
      DØ 326 I = 1, 63
  326 R3G5(I) = 0.0
      K = 1
      DØ 327 I = 1, 44
      DØ 328 J = 1, 20
      R3G5(K) = R3G5(K) + (R3B(I)*G5B(J))
```

```
 328 K = K + 1
 327 K = I + 1
     DØ 329 I = 1, 20
 329 G45(I) = G4B(I) + G5BP(I)
     DØ 330 I = 1, 63
 330 R445G(I) = 0.0
     K = 1
     DØ 331 I = 1, 44
     DØ 332 J = 1, 20
     R445G(K) = R445G(K) + (R4B(I)*G45(J))
 332 K = K + 1
 331 K = I + 1
     DØ 333 I = 1, 63
 333 T5(I) = R1F5(I) + R245Z(I) + R3G5(I) + R445G(I)
     WRITE ØUTPUT TAPE 6, 430
     WRITE ØUTPUT TAPE 6, 405, T5
C    T6
     DØ 334 I = 1, 63
 334 R1F4(I) = 0.0
     K = 1
     DØ 335 I = 1, 48
     DØ 336 J = 1, 16
     R1F4(K) = R1F4(K) + (R1B(I)*F4B(J))
 336 K = K + 1
```

```
335 K = I + 1
    DØ 337 I = 1, 63
337 R234Z(I) = 0.0
    K = 1
    DØ 338 I = 1, 48
    DØ 339 J = 1, 16
    R234Z(K) = R234Z(K) + (R2B(I)*F34(J))
339 K = K + 1
338 K = I + 1
    DØ 340 I = 1, 63
340 R3G4(I) = 0.0
    K = 1
    DØ 341 I = 1, 44
    DØ 342 J = 1, 20
    R3G4(K) = R3G4(K) + (R3B(I)*G4B(J))
342 K = K + 1
341 K = I + 1
    DØ 343 I = 1, 20
343 G34(I) = G3B(I) + G4BP(I)
    DØ 344 I = 1, 63
344 R434G(I) = 0.0
    K = 1
    DØ 345 I = 1, 44
    DØ 346 J = 1, 20
```

```
      R434G(K) = R434G(K) + (R4B(I)*G34(J))
  346 K = K + 1
  345 K = I + 1
      DØ 347 I = 1, 63
  347 T6(I) = R1F4(I) + R234Z(I) + R3G4(I) + R434G(I)
      WRITE ØUTPUT TAPE 6, 431
      WRITE ØUTPUT TAPE 6, 405, T6
C     T7
      DØ 348 I = 1, 63
  348 R1F3(I) = 0.0
      K = 1
      DØ 349 I = 1, 48
      DØ 350 J = 1, 16
      R1F3(K) = R1F3(K) + (R1B(I)*F3B(J))
  350 K = K + 1
  349 K = I + 1
      DØ 351 I = 1, 63
  351 R223Z(I) = 0.0
      K = 1
      DØ 352 I = 1, 48
      DØ 353 J = 1, 16
      R223Z(K) = R223Z(K) + (R2B(I)*F23(J))
  353 K = K + 1
  352 K = I + 1
```

```
      DØ 654 I = 1, 63
  654 R3G3(I) = 0.0
      K = 1
      DØ 354 I = 1, 44
      DØ 355 J = 1, 20
      R3G3(K) = R3G3(K) + (R3B(I)*G3B(J))
  355 K = K + 1
  354 K = I + 1
      DØ 356 I = 1, 20
  356 G23(I) = G2B(I) + G3BP(I)
      DØ 357 I = 1, 63
  357 R423G(I) = 0.0
      K = 1
      DØ 358 I = 1, 44
      DØ 359 J = 1, 20
      R423G(K) = R423G(K) + (R4B(I)*G23(J))
  359 K = K + 1
  358 K = I + 1
      DØ 360 I = 1, 63
  360 T7(I) = R1F3(I) + R223Z(I) + R3G3(I) + R423G(I)
      WRITE ØUTPUT TAPE 6, 432
      WRITE ØUTPUT TAPE 6, 405, T7
C     T8
      DØ 361 I = 1, 63
```

```
361  R1F2(I) = 0. 0
     K = 1
     DØ 362 I = 1, 48
     DØ 363 J = 1, 16
     R1F2(K) = R1F2(K) + (R1B(I)*F2B(J))
363  K = K + 1
362  K = I + 1
     DØ 364 I = 1, 63
364  R212Z(I) = 0. 0
     K = 1
     DØ 365 I = 1, 48
     DØ 366 J = 1, 16
     R212Z(K) = R212Z(K) + (R2B(I)*F12(J))
366  K = K + 1
365  K = I + 1
     DØ 367 I = 1, 63
367  R3G2(I)   0. 0
     K = 1
     DØ 368 I = 1, 44
     DØ 369 J = 1, 20
     R3G2(K) = R3G2(K) + (R3B(I)*G2B(J))
369  K = K + 1
368  K = I + 1
     DØ 370 I = 1, 20
```

```
370 G12(I) = G1B(I) + G2BP(I)
    DØ 371 I = 1, 63
371 R412G(I) = 0.0
    K = 1
    DØ 371 I = 1, 44
    DØ 373 J = 1, 20
    R412G(K) = R412G(K) + (R4B(I)*G12(J))
373 K = K + 1
372 K = I + 1
    DØ 374 I = 1, 63
374 T8(I) = R1F2(I) + R212Z(I) + R3G2(I) + R412G(I)
    WRITE ØUTPUT TAPE 6, 433
    WRITE ØUTPUT TAPE 6, 405, T8
C   T9
    DØ 375 I = 1, 63
375 R1F1(I) = 0.0
    K = 1
    DØ 376 I = 1, 48
    DØ 377 J = 1, 16
    R1F1(K) = R1F1(K) + (R1B(I)*F1B(J))
377 K = K + 1
376 K = I + 1
    DØ 378 I = 1, 63
378 R2F1P(I) = 0.0
```

```
      K = 1
      DØ 379 I = 1, 48
      DØ 380 J = 1, 16
      R2F1P(K) = R2F1P(K) + (R2B(I)*F1BP(J))
  380 K = K + 1
  379 K = I + 1
      DØ 381 I = 1, 63
  381 R3G1(I) = 0.0
      K = 1
      DØ 382 I = 1, 44
      DØ 383 J = 1, 20
      R3G1(K) = R3G1(K) + (R3B(I)*G1B(J))
  383 K = K + 1
  382 K = I + 1
      DØ 384 I = 1, 63
  384 R4G1P(I) = 0.0
      K = 1
      DØ 385 I = 1, 44
      DØ 386 J = 1, 20
      R4G1P(K) = R4G1P(K) + (R4B(I)*G1BP(J))
  386 K = K + 1
  385 K = I + 1
      DØ 387 I = 1, 63
  387 T9(I) = R1F1(I) + R2F1P(I) + R3G1(I) + R4G1P(I)
```

```
      WRITE ØUTPUT TAPE 6, 434
      WRITE ØUTPUT TAPE 6, 405, T9
      CALL EXIT
    1 FØRMAT (4E18.7)
  402 FØRMAT (1H1(4E18.7))
  404 FØRMAT (15H A1 THRØUGH A17)
  405 FØRMAT (1H (6E18.7))
  407 FØRMAT (15H B1 THRØUGH B15)
  408 FØRMAT (15H C1 THRØUGH C20)
  410 FØRMAT (14H1D1 THRØUGH D9)
  413 FØRMAT (15H E1 THRØUGH E11)
  414 FØRMAT (14H D1B, D2B, D3B)
  415 FØRMAT (16H E1B THRØUGH E7B)
  416 FØRMAT (16H F1B THRØUGH F7B)
  417 FØRMAT (10H1R1 AND R2)
  418 FØRMAT (11H Y1, Y2, Y3)
  420 FØRMAT (16H R1B THRØUGH R4B)
  423 FØRMAT (16H1G1B  THRØUGH G8B)
  424 FØRMAT (3H T1)
  426 FØRMAT (3H T2)
  427 FØRMAT (3H1T3)
  429 FØRMAT (3H T4)
  430 FØRMAT (3H T5)
  431 FØRMAT (3H T6)
```

```
432 FØRMAT (3H1T7)
433 FØRMAT (3H T8)
434 FØRMAT (3H T9)
    END(1,0,0,0,0,0,0,0,0,0,0,0,0,0,0)
```

APPENDIX C

24843

AN ASYMPTOTIC SOLUTION FOR CONICAL SHELLS OF LINEARLY VARYING THICKNESS

Prepared by

Chin Hao Chang

This appendix was previously submitted as Technical Report C for contract NAS 8-5168

ABSTRACT

24843

An asympotic general solution of a segment of an elastic and isotropic truncated conical shell with linearly varying thickness subjected to lateral normal loads is presented. The segment is free from normal force and moment along the two straight edges but arbitrarily supported along the two circular ends. As an example, a solution is given for a segment with one circular end free and the other end fixed.

author

Technical Report C for NASA Contract NAS8-5168

AN ASYMPTOTIC SOLUTION FOR CONICAL SHELLS OF LINEARLY VARYING THICKNESS

Prepared By

Chin Hao Chang, Ph. D.+

I. INTRODUCTION

The results presented in this report are a continuation of the work which has been reported in a Summary Report of November, 1962, for Contract No. NAS8-5012, see Reference (1).

In that report, a general method leading to the solution of the problem of a segment of conical shell subjected to laternal normal load was given. An exact solution could be obtained by following that method. However, the numerical computations involved would be very laborious. A closer study of the data for an engine shroud supplied by the sponsoring agency has revealed that the bending effect is almost negligible. In other words, many terms associated with the parameter k which accounts for the bending effect are not necessary. Since the shell itself is quite thin, k is a very small value. Because of this fact, a limiting case as k approaches zero asymptotically has been investigated. An asymptotic solution for this case has been obtained.

In this report, for completeness, the entire problem and its basic formulations are given. A method for obtaining the homogeneous solution follows and then a general asymptotic solution of a segment of conical shell subjected to a lateral normal load is presented. As an example of application of the general solution, the solution for a cantilever segment with the data supplied by the sponsoring agency is

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given along with numerical curves directly applicable for design purpose.

II. THE BASIC FORMULATIONS

Consider a segment of a truncated thin conical shell of elastic isotropic and homogeneous material. The middle surface of the shell is described by the co-ordinates s and θ, where s is the distance to a point measured from the apex along a generator and θ is the angle measured from an edge meridian to the generator on which the point lies. The inclination of s with respect to the central axis in indicated by an angle α the complement of the half-central angle of the eone. The thickness t of the shell is assumed to be in direct proprotion to the distance s, i.e.,

$$t=\delta s \tag{1}$$

where δ is a constant of proportionality. The segment is bounded by $s = L_1, L$ and $\theta=0$ and θ_1 as shown in Fig. (1).

Let u, v, and w be the three displacement components in the s, θ, and normal-to-the-middle-surface directions respectively. The elastic law assumes the following relationships between the forces and displacements:

$$N_s = D[V^{\cdot} + \frac{\nu}{s}(U'\sec\alpha + v + w\tan\alpha)] - K\frac{w^{\cdot\cdot}}{s}\tan\alpha ,$$

$$N_\theta = D[\frac{1}{s}(\frac{U'}{\cos\alpha} + V + W\tan\alpha) + \nu V^{\cdot}]$$

$$+ K\frac{1}{s^3}\cdot[\nu\tan\alpha + w\tan^2\alpha + w''\sec^2\alpha + sw^{\cdot}]\tan\alpha ,$$

$$N_{s\theta} = D\frac{1-\nu}{2}[u^{\cdot} - \frac{u}{s} + \frac{v'}{s\cos\alpha}]$$

$$+ K\frac{1-\nu}{2}\frac{1}{s^3}[su^{\cdot} - u - sw'^{\cdot}\frac{1}{\sin\alpha} + w'\frac{1}{\sin\alpha}]\tan^2\alpha ,$$

$$N_{\theta s} = D\,\frac{1-\nu}{2}\,\frac{1}{s}\left[\, su^{\cdot} - u + v' \sec\alpha \,\right] \qquad (2)$$

$$+ K\,\frac{1-\nu}{2}\,\frac{1}{s^3}\left[V' \sec\alpha + sw'^{\cdot}\,\frac{1}{\sin\alpha} - w'\,\frac{1}{\sin\alpha}\right] \tan^2\alpha\,,$$

$$M_s = K\,\frac{1}{s^2}\left[\, s^2 w^{\cdot\cdot} - sv^{\cdot}\tan\alpha + \nu(w''\sec^2\alpha + sw^{\cdot} - u'\sec\alpha\tan\alpha)\right],$$

$$M_\theta = K\,\frac{1}{s^2}\left[W'' \sec^2\alpha + sw^{\cdot} + w\tan^2\alpha + \nu s^2 w^{\cdot\cdot}\right],$$

$$M_{s\theta} = K(1-\nu)\,\frac{1}{s^2}\left[\, sw'^{\cdot} - w' - su^{\cdot}\sin\alpha + u\sin\alpha \,\right] \sec\alpha\,,$$

$$M_{\theta s} = K\,(1-\nu)\,\frac{1}{s^2}\left[\, sw'^{\cdot} - w' - \tfrac{1}{2}\,su^{\cdot}\sin\alpha + \tfrac{1}{2}\,u\sin\alpha + \tfrac{1}{2}\,v'\tan\alpha\right] \sec\alpha\,,$$

in which N_s, N_θ, M_s, M_θ, are normal forces and moments per unit length in the s and θ directions; $N_{s\theta}$ and $M_{s\theta}$ are shearing forces and twisting moment per unit length on a section normal to s and acting in the θ direction, and the constants D and K are defined as following:

$$D = \frac{Et}{1-\nu^2}\,, \quad K = \frac{Et^3}{12(1-\nu^2)}\,. \qquad (3)$$

The dots indicate partial differentiation with respect to s and primes differentiation with respect to θ.

Let P_s, P_θ, and P_r be the load components per unit area in the S, θ, and normal-to-the-surface directions respectively. The six equations of equilibrium assume the following form :

$$\begin{aligned}
&(SN_s)^{\cdot} + N'_{\theta s} \sec\alpha - N_\theta = -P_s S \\
&S(N_{s\theta})^{\cdot} + N'_\theta \sec\alpha + N_{\theta s} - Q_\theta \tan\alpha = -P_\theta S \\
&N_\theta \tan\alpha + Q'_\theta \sec\alpha + (S\theta_s)^{\cdot} = P_r S \\
&(SM_s)^{\cdot} + M'_{\theta s} \sec\alpha - M_\theta = SQ_s \\
&(SM_{s\theta})^{\cdot} + M'_\theta \sec\alpha + M_{\theta s} = SQ_\theta \\
&S(N_{\theta s} - N_{s\theta}) = M_{\theta s} \tan\alpha
\end{aligned} \qquad (4)$$

where Q_θ and Q_s are the transverse shearing forces acting on θ and s planes respectively.

The last of equations (4) is an identity which can be seen by making use of stress-displacement law (2). Therefore, this equation may be dropped. Using the third and fourth equations of equations (4), the transverse shearing forces Q_θ and Q_s may be eliminated from the other three. Finally there are three equations of equilibrium:

$$
\begin{aligned}
&(SN_s)^{\cdot} + N'_{\theta s} \sec\alpha - N_\theta = -P_s S, \\
&SN_\theta \tan\alpha + S(SM_s)^{\cdot\cdot} + (SM'_{s\theta})^{\cdot} \sec\alpha + (SM'_{\theta s})^{\cdot} \sec\alpha \\
&+ M''_\theta \sec^2\alpha - SM^{\cdot}_\theta = P_r S^2. \\
&S(SN_{s\theta})^{\cdot} + SN'_\theta \sec\alpha + SN_{\theta s} - (SM_{s\theta})^{\cdot} \tan\alpha \\
&-M_{\theta s} \tan\alpha - M'_\theta \tan\alpha \sec\alpha = -P_\theta S^2,
\end{aligned}
\tag{5}
$$

These three equations may be expressed by means of the elastic law in terms of the three displacements in the following form:

$$
\begin{aligned}
&\frac{1-\nu}{2} S^2 u^{\cdot\cdot} + u'' \sec^2\alpha + (1-\nu)\, su^{\cdot} - (1-\nu)u + \frac{1+\nu}{2}\, sv'^{\cdot} \sec\alpha \\
&+ (2-\nu)\, V' \sec\alpha + s' \tan\alpha \sec\alpha + k\Big[\frac{3}{2}(1-\nu)\, s^2 v^{\cdot\cdot} \tan\alpha \\
&+ 3(1-\nu)\, su^{\cdot} \tan\alpha - 3(1-\nu)\, u \tan\alpha - \Big(\frac{3-\nu}{2}\Big) s^2 w'^{\cdot\cdot} \sec\alpha \\
&-3(1-\nu)\, sw'^{\cdot} \sec\alpha + 3(1-\nu)\, w' \sec\alpha\Big] \tan\alpha = -\frac{P_\theta s^2}{D}
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
&\frac{(1+\nu)}{2}\, su'^{\cdot} \sec\alpha - \frac{3}{2}(1-\nu)\, u' \sec\alpha + s^2 v^{\cdot\cdot} + \frac{1-\nu}{2}\, v'' \sec\alpha \\
&+2Sv^{\cdot} - (1-\nu)v + \nu\, sw^{\cdot} \tan\alpha - (1-\nu)\, w \tan\alpha \\
&+k\Big[\frac{1-\nu}{2}\, v'' \tan\alpha \sec^2\alpha - v \tan\alpha - s^3 w^{\cdot\cdot\cdot} + \frac{1-\nu}{2}\, sw''^{\cdot} \sec^2\alpha \\
&- 3S^2 w^{\cdot\cdot} - \frac{3-\nu}{2}\, w'' \sec^2\alpha - sw^{\cdot} - w \tan^2\alpha\Big] \tan\alpha = \frac{-P_s S^2}{D}
\end{aligned}
$$

$$[u' \sec\alpha + \nu s v^{\cdot} + v + w\tan\alpha]\tan\alpha + k\Big[-\frac{3-\nu}{2}s^2u'^{\cdot\cdot}\sec\alpha$$

$$-(3+\nu)su'^{\cdot}\sec\alpha + (3-5\nu)u'\sec\alpha - s^3v^{\cdot\cdot\cdot} + \frac{1-\nu}{2}sv''^{\cdot}\sec^2\alpha$$

$$-6s^2v^{\cdot\cdot} + (2-\nu)V''\sec^2\alpha - 7SV^{\cdot} - v(1-\tan^2\alpha)\Big]\tan\alpha$$

$$+k\Big[s^4w^{\cdot\cdot\cdot\cdot} + 2s^2w''^{\cdot\cdot}\sec^2\alpha + w''''\sec^4\alpha + 8S^3 w^{\cdot\cdot\cdot}$$
$$+4SW''^{\cdot}\sec^2\alpha + s^2(11+3\nu)W^{\cdot\cdot} + 2W''\tan^2\alpha\sec^2\alpha$$
$$-(5-6\nu)w''\sec^2\alpha - 2(1-3\nu)sw^{\cdot} - w(1-\tan^2\alpha)\tan^2\alpha\Big] = P_r\frac{s^2}{D}$$

where $$k = \frac{\delta^2}{12} \qquad (6a)$$

The above formulations of the problem are given in Reference (2). They are exact in a sense that only terms of second and higher power of K or equivalent of k are neglected in the elastic law (2).

The segments considered are free from normal moment and force along the two straight edges such that

$$M_\theta = 0 \text{ and } N_\theta = 0 \text{ for } \theta = 0 \text{ and } \theta_1 \qquad (7)$$

These two conditions are satisfied by assuming

$$u = \sum_{n=1} A_n f_n(s) \cos\frac{n\pi\theta}{\theta_1}$$

$$V = \sum_{n=1} B_n f_n(s) \sin\frac{n\pi\theta}{\theta_1}$$

$$W = \sum_{n=1} C_n f_n(s) \sin\frac{n\pi\theta}{\theta_1} \qquad (8)$$

where A_n, B_n, and C_n are constants and $f_n(s)$ is a function of s to be determined by the set of equations (6). Now consider the case in which the segment is subjected to only a lateral normal load. Thus

$$P_\theta = P_s = 0$$

and let

$$P_r = \sum_{n=1} A_n P_n(s) \sin\frac{n\pi\theta}{\theta_1} \qquad (9)$$

For convenience, a non-dimensional variable is introduced such that

$$y = \sqrt{\frac{s}{L}} \tag{10}$$

along with the assumption

$$f_n(s) = y^{\lambda_n - 1} \tag{11}$$

in which λ_n are constants to be determined. Substituting equations (8) to (11) into (6) results

$$\begin{aligned} d_{11}A_n + d_{12}B_n + d_{13}C_n &= 0 \\ d_{21}A_n + d_{22}B_n + d_{23}C_n &= 0 \\ d_{31}A_n + d_{32}B_n + d_{33}C_n &= \frac{(1-\nu^2)L}{E\,\delta} A_n P_n(y) y^{3-\lambda_n} \end{aligned} \tag{12}$$

where

$$\begin{aligned} d_{11} &= \frac{1-\nu}{8}(1+3k\tan^2\alpha)(9-\lambda_n^2) + m_n^2 \\ d_{12} &= \frac{1}{4}\left[(7-5\nu) + (1+\nu)\lambda_n\right] m_n \\ d_{13} &= \left\{1 + \frac{k}{8}\left[3(9-11\nu) + 8\nu\lambda_n - (3-\nu)\lambda_n^2\right]\right\} m_n \tan\alpha \\ d_{22} &= \frac{1}{4}(1-\lambda_n^2) + (1-\nu)(1+\tfrac{1}{2}m_n^2) + k\tan^2\alpha\,(1+\frac{1-\nu}{2}m_n^2) \\ d_{23} &= \frac{1}{2}\tan\alpha\left[(2-\nu) - \nu\lambda_n\right] - \frac{1}{8}k\tan\alpha\left\{1-8\tan^2\alpha\right. \\ &\quad \left. + 2(7-3)m_n^2 - 3[3+2(1-\nu)m_n^2]\lambda_n + 3\lambda_n^2 - \lambda_n^3\right\} \\ d_{33} &= \tan^2\alpha + \frac{1}{16}k\left\{(13-12\nu) - 16(1-\tan^2\alpha)\tan^2\alpha\right. \\ &\quad \left. + 8\left[(11-12\ ) - 4\tan^2\alpha\right] m_n^2 + 16m_n^4 - 2\left[(7-6\nu)+4m_n^2\right]\lambda_n^2 + \lambda_n^4\right\} \end{aligned} \tag{13}$$

while d_{21}, d_{32}, and d_{31} are obtained by replacing λ_n by $-\lambda_n$ in d_{12} and d_{23} and d_{13} respectively and where

$$m_n \equiv \frac{n\pi}{\theta_1}\sec\alpha \tag{14}$$

III. ON THE HOMOGENEOUS SOLUTIONS

The homogeneous solutions are obtained from the following three simultaneous equations of equations (12)

$$d_{11}A + d_{12}B + d_{13}C = 0$$

$$d_{21}A + d_{22}B + d_{23}C = 0$$

$$d_{31}A + d_{32}B + d_{33}C = 0 \qquad (15)$$

in which and also hereafter the subscript n is omitted for brevity. For the constants A, B and C to be non-trival, the determinant must vanish i.e.

$$\begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix} = 0 . \qquad (16)$$

Substituting the coefficients given by (13) into the determinant and neglecting the terms of second and higher power of k, as one did in obtaining the elastic law (2), yields,

$$k[\lambda^8 - g_6\lambda^6 + g_4\lambda^4 - g_2\lambda^2 + g_0] + G[\lambda^4 - 10\lambda^2 + 9] = 0 \qquad (17)$$

where

$$G = 16(1-\nu^2)\tan^2\alpha$$

$$g_6 = 4(7-4\nu) - 8\tan^2\alpha + 16m^2$$

$$g_4 = 2[(127-136\nu + 24\nu^2) - 4(8+3\nu)\tan^2\alpha + 8(4-3\nu^2)\tan^4\alpha]$$
$$+ 16[17-12\nu-6\tan^2\alpha]\,m^2 - 96m^4 \qquad (18)$$

$$g_2 = 4[(203-216 + 120\nu^2) - 2(80-61\nu)\tan^2\alpha + 40(4-3\nu^2)\tan^2\alpha]$$
$$+16[(71-72\nu) - 4(13-10\nu)\tan^2\alpha + 8(2-\nu)\tan^4\alpha]m^2$$
$$+ 64[(13-12\nu) - 2(4-\nu)\tan^2\alpha]\,m^4 + 256m^6$$

$$g_0 = 9[(13-12\nu)(5-4\nu) - 8(8-7\nu)\tan^2\alpha + 16(4-3\nu^2)\tan^4\alpha]$$
$$+ 16[(215-412\nu+192\nu^2) + 2(89-172\nu+96\nu^2)\tan^2\alpha$$
$$+ 40(2-\nu)\tan^4\alpha]m^2$$
$$- 32[(81-184\nu+96\nu^2) + 4(16-13\ )\tan^2\alpha - 8\tan^2\alpha]m^4$$
$$+ 256[(3-4\nu) - 2\tan^2\alpha]\,m^6 + 256\,m^8$$

The equation (17) could be solved by Brown's method as was mentioned in the Summary Report. However, it has been found that this method is quite laborious. In view of the approximations made in arriving at the equation itself, a method which, to some extent is similar to so called perturbation method is used. This method gives a quick result with the same accuracy provided by the present theory.

Introducing

$$\lambda^2 = x_0 + kx_1 + \ldots \tag{19}$$

into equation (17) and neglecting the terms of second and higher power of k gives

$$\begin{aligned} k[x_0^4 - g_6 x_0^3 + g_4 x_0^2 - g_2 x_0 + g_0 + G(2x_0 x_1 - 10x_1)] \\ + G[x_0^2 - 10x_0 + 9] = 0\,. \end{aligned} \tag{20}$$

This equation is satisfied if

$$x_0^2 - 10x_0 + 9 = 0 \tag{21}$$

and

$$x_0^4 - g_6 x_0^3 + g_4 x_0^2 - g_2 x_0 + g_0 + 2G(x_0 - 5)\, x_1 = 0\,. \tag{22}$$

Equation (21) provides two roots of x_0

$$x_0 = 1 \text{ and } 9\,. \tag{23}$$

Solving for x_1 from equation (22)

$$x_1 = -\frac{x_0^4 - g_6 x_0^3 + g_4 x_0^3 - g_2 x_0 + g_0}{2G(x_0 - 5)} \tag{24}$$

Thus two roots of λ^2 are obtained. Denoting them by $\lambda_1{}^2$ and $\lambda_2{}^2$,

$$\begin{aligned} \lambda_1{}^2 &= 1 + k\,\frac{1 - g_6 + g_4 - g_2 + g_0}{8G} \\ \lambda_2{}^2 &= 9 - k\,\frac{9^4 - 9^3 g_6 + 9^2 g_4 - 9g_2 + g_0}{8G} \end{aligned} \tag{25}$$

Let the other two roots of λ^2 be $\lambda_3{}^2$ and $\lambda_4{}^2$. Then equation (17) may be written as follows:

$$(\lambda^2 - \lambda_1{}^2)(\lambda^2 - \lambda_2{}^2)(\lambda^2 - \lambda_3{}^2)(\lambda^2 - \lambda_4{}^2) = 0$$

Expanding the previous equation and equating the coefficients of λ^6 and λ^0 to the corresponding ones in equation (17), one obtains

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = g_6$$

$$\lambda_1^2 \lambda_2^2 \lambda_3^2 \lambda_4^2 = g_0 + \frac{9G}{k}$$

Solving for λ_3^2 and λ_4^2, yields

$$\lambda_{\substack{3\\4}}^2 = \tfrac{1}{2}(g_6 - \lambda_2^2 - \lambda_1^2) \pm i\sqrt{\frac{1}{\lambda_1^2\lambda_2^2}(g_0 + \frac{9G}{k}) - [\tfrac{1}{2}(g_6 - \lambda_1^2 - \lambda_2^2)]} \tag{26}$$

Finally four real roots of λ are obtained from equation (25) and four complex roots from (26).

With the eight roots computed, one may follow the method outlined in the Summary Report to obtain the homogeneous solutions of the displacements. However, it would be almost impossible to work out the solutions in a general form. A closer study of the algebric expressions of the solutions reveals that many terms are negligible because of smallness of the parameter k (see Table 1). If only the terms with the lowest power of k are retained in the algebric expressions, the solution will be simplified. Such simplified solutions would be adequate for practical design of thin shells such as the one being concerned. In what follows, a solution considering the parametric constant k approaching zero asymptotically is presented.

IV. THE ASYMPTOTIC SOLUTIONS

(1). The Asymptotic Homogeneous Solutions

Let the parameter k approach zero asymptotically and only the terms with the lowest power of k be retained, then the eight roots of λ obtained from equations (25) and (26) become

$$\lambda_{\substack{1\\2}} = \pm 1, \quad \lambda_{\substack{3\\4}} = \pm 3 \tag{27}$$

$$\lambda_{\substack{5\\6}} = \pm\rho(1+i), \quad \lambda_{\substack{7\\8}} = \pm\rho(1-i) \tag{28}$$

where

$$\rho \equiv \left| \frac{\sqrt{2}}{2} \left(\frac{G}{k} \right)^{1/4} \right| \tag{29}$$

When $\lambda = \lambda_i$ (i = 1,2,3, and 4) the corresponding coefficients A_i and B_i may be solved in terms of C_i. From the first two of equations (15),

$$A_i = -\frac{4m \tan\alpha}{M} \left[2(\lambda_i^2+\lambda_i+2)+(\lambda_i+1)(\lambda_i-5)\nu - 4m^2 \right] C_i \tag{30}$$

$$B_i = -\frac{2 \tan\alpha}{M} (\lambda_i - 3) \left\{ (\lambda_i+3)\left[(\lambda_i+1)\nu - 2 \right] + 4m^2 \right\} C_i$$

where

$$M \equiv (\lambda_i^2-9)(\lambda_i^2-5+4\nu) + 8m^2(2m^2+4\nu-5-\lambda_i^2).$$

Substituting the values of λ_i given by (27) into the foregoing formulation yields a set of the homogeneous solutions. Denoting them by U_I, V_I and W_I, they are

$$U_I = m \tan\alpha \left\{ \frac{1}{m^2-1} C_1 + \frac{C_2}{m^2-2(1-\nu)} y^{-2} + \frac{C_3}{m^2} y^2 + \frac{4+4\nu-m^2}{m^2[7-2\nu-m^2]} C_4 y^{-4} \right\} \cos\frac{n\pi\theta}{\theta_1}$$

$$V_I = \tan\alpha \left\{ \frac{1}{m^2-1} C_1 + \frac{2C_2}{m^2-2(1-\nu)} y^{-2} + \frac{3C_4}{m^2-7+2\nu} y^{-4} \right\} \sin\frac{n\pi\theta}{\theta_1} \tag{31}$$

$$W_I = \left\{ C_1 + C_2 y^{-2} + C_3 y^2 + C_4 y^{-4} \right\} \sin\frac{n\pi\theta}{\theta_1}$$

When $\lambda = \lambda_j$ (j=5,6,7 and 8)

$$\overline{A_j} = 4m \tan\alpha \left[-(2+\nu)\frac{1}{\lambda_j^2} \right] \overline{C_j} \tag{32}$$

$$\overline{B_j} = -2\nu \tan\alpha \frac{1}{\lambda_j} \overline{C_j}$$

which are obtained again from the first two of equations (15). The bars introduced are simply to indicate that these coefficients are complex numbers. To present the resulting solutions in a real form, let

$$\overline{C_5} = \frac{1}{2}(C_5 - iC_6) \qquad \overline{C_7} = \frac{1}{2}(C_7 - iC_8)$$

$$\overline{C_6} = \frac{1}{2}(C_5 + iC_8) \qquad \overline{C_8} = \frac{1}{2}(C_7 + iC_8) \tag{33}$$

and apply the identity

$$y^{i\rho} = e^{i\rho \ell ny} = \cos(\rho \ell ny) + i \sin(\rho \ell ny)$$

to these solutions which denoted by U_{II}, V_{II} and W_{II} become

$$U_{II} = m\,(2+\nu)\tan\alpha\left(\frac{k}{G}\right)^{1/2} y^{-1}\Big\{y^{\rho}[C_6 \cos(\rho \ell ny) - C_5 \sin(\rho \ell ny)]$$

$$+\, y^{-\rho}[-C_8 \cos(\rho \ell ny) + C_7 \sin(\rho \ell ny)]\Big\} \cos\frac{n\pi\theta}{\theta_1}$$

$$V_{II} = -\sqrt{2}\,\nu \tan\alpha\left(\frac{k}{G}\right)^{1/4} y^{-1}\Big\{ y^{\rho}[(C_5-C_6)\cos(\rho \ell ny) + (C_5+C_6)\sin(\rho \ell ny)]$$

$$+\, y^{-\rho}[-(C_7-C_8)\cos(\rho \ell ny) + (C_7+C_8)\sin(\rho \ell ny)]\Big\} \frac{n\pi\theta}{\theta_1}$$

$$W_{II} = y^{-1}\Big\{ y\,[C_5\cos(\rho \ell ny) + C_6 \sin(\rho \ell ny)$$

$$+\, y^{-\rho}[C_7\cos(\rho \ell ny) + C_8\sin(\rho \ell ny)]\Big\} \sin\frac{n\pi\theta}{\theta_1} \qquad (34)$$

In both equations (31) and (34) the summation signs are omitted.

The general homogeneous solutions consist of both sets of solutions (31) and (34). The eight constants C_1, C_2, ... C_8 are to be determined by the eight prescribed boundary conditions including edge loads at the two circular ends.

The first set of solutions are simply the solutions of membrane theory. In other words, when λ is a finite constant as k approaches zero, the equations of membrane theory may be used for asymptotic solutions. Based on this fact, the equations of membrane theory will be used for the asymptotic particular solutions of the problem.

(2) Particular Solutions

The lateral normal load defined in expression (9) is confined in such a form that

$$P_n(s) = S^r$$

or

$$P_n(y) = L^r y^{2r} \qquad (35)$$

where r is a given constant.

Substituting expression (35) into equations (12) and equating

$$\lambda = \lambda^* = 2r+3 ,$$

the particular solutions are readily obtained by solving the three algebric simultaneous equations (12) provided that λ^* is not equal to any one of the λ_i or λ_j, the roots of the determinant. When λ^* does equal one of the roots, a particular approach must be used which will be demonstrated in what follows.

Consider a case when the load is uniformly distributed in the s-direction. (This is the load specified by the sponsoring agency for the engine shroud.) In this case, P_n is unity, r = 0 and $\lambda^* = 3$ which is one of the roots of the determinant of the asymptotic homogeneous solutions given by (27). Since λ^* is a finite constant, the corresponding particular solutions may be obtained from the equations of membrane theory. These equations are obtained simply by letting k = 0 in equation (6).

Assume a set of particular solutions denoted by U_p, V_p and W_p in the form:

$$U_p = \sum_{n=1} (d_{n1} + d_{n2}\, \ell ny)\, y^2 \cos \frac{n\pi\theta}{\theta_1}$$

$$V_p = \sum_{n=1} (b_{n1} + b_{n2}\, \ell ny)\, y^2 \sin \frac{n\pi\theta}{\theta_1}$$

$$W_p = \sum_{n=1} C_{n1}(1+\ell ny)\, y^2 \sin \frac{n\pi\theta}{\theta_1} \qquad (36)$$

and put

$$P_r = \sum_{n=1} a_n \sin \frac{n\pi\theta}{\theta_1}$$

$$P_\theta = P_s = 0$$

where $y = \sqrt{\frac{S}{L}}$ as before, d_{n1}, d_{n2}, b_{n1}, b_{n2} and C_{n1} are constants to be determined.

Setting k = 0 and introducing functions (36) into equation (6), after cancelling out the sinusoidal functions, the following equations are obtained:

$$m[-md_2 + \frac{1}{2}(5-\nu)\, b_2 + \tan\alpha\, C_1]\ \ln y + \{m\,[-md_1 + \frac{1}{2}(5-\nu)\, b_1 + \tan\alpha\, C_1] + \frac{3}{4}(1-\nu)d_2 + \frac{1}{4}(1+\nu)\, mb_2\} = 0$$

$$[m(1-2\nu)\, d_2 + (1+\nu-\frac{1-\nu}{2}m^2)\, b_2 + (2\nu-1)\tan\alpha\, C_1]\ \ln y + [m(1-2\nu)d_1 + (1+\nu-\frac{1-\nu}{2}m^2)\, b_1 + \frac{1}{2}(5\nu-2)\tan\alpha\, C_1 - \frac{1+\nu}{4}md_2 + \frac{3}{2}b_2] = 0 \qquad (37)$$

$$[-md_2+(1+\nu)b_2+\tan\alpha\, C_1]\ln y+[-md_1+(1+\nu)b_1+\tan\alpha\, C_1 + \frac{1}{2}\nu b_2] = \frac{aL}{\tan\alpha}\ \frac{12(1-\nu^2)}{E\delta}.$$

The subscripts n have been dropped for simplicity.

Equations (37) are satisfied if the coefficients of the independent variable y on the both sides of equations (37) are equal. Two sets of equations may be obtained for the constants: The first set is obtained by equating the coefficients of lny,

$$m[-md_2+\frac{1}{2}(5-\nu)b_2 + \tan\alpha\, C_1] = 0$$

$$[m(1-2\nu)d_2 + (1+\nu-\frac{1-\nu}{2}m^2)\, b_2 + (2\nu-1)\tan\alpha\, C_1] = 0 \qquad (38)$$

$$-ma_2 + (1+\nu)\, b_2 + \tan\alpha\, C_1 = 0$$

and the second set is obatined by equating the coefficients of y^0

$$m[-md_1 + \frac{1}{2}(5-\nu)\, b_1 + \tan\alpha\, C_1] + \frac{3}{4}(1-\nu)\, d_2+\frac{1}{4}(1+\nu)mb_2 = 0$$

$$m(1-2\nu)d_1 + (1+\nu-\frac{1-\nu}{2}m^2)b_1 + \frac{1}{2}(5\nu-2)\tan\alpha\, C_1 - \frac{1+\nu}{4}md_2 + \frac{3}{2}b_2 = 0 \qquad (39)$$

$$-md_1+(1+\nu)b_1 + \tan\alpha\, C_1 + \frac{1}{2}\nu b_2 = \frac{aL}{\tan\alpha}\ \frac{12(1-\nu^2)}{E}$$

Note that equations (38) are the same as equations (15) when k = 0 and $\lambda = \lambda_3 = 3$. Since $\lambda = 3$ is a root of the determinent of equations (38), there are only two independent equations in equations (38). Solving for the five constants d_1, d_2, b_1, b_2, and C_1, from the five independent equations of (38) and (39), the results are:

$$U_p = \frac{L}{E\delta}\ \frac{1}{\tan\alpha} \sum_n a_n \frac{m_n}{3} \frac{1}{2m_n^2}\,[2m_n^4 - 3(5-\nu)\, m_n^2 - 3(1+\nu)] + (m_n^2 - 7 + 2\nu)\ \ln y\}\ y^2 \cos\frac{n\pi\theta}{\theta_1} \qquad (40)$$

$$V_p = \frac{L}{E\delta}\frac{1}{\tan\alpha}\ \frac{1}{6}\sum_n a_n[3(1-2\nu) - m_n{}^2]y^2 \sin\frac{n\pi\theta}{\theta_1}$$

$$W_p = \frac{1}{E\delta}\frac{1}{\tan^2\alpha}\ \frac{1}{3}\sum_n m_n[m_n{}^2-7+2\nu]\ (1+\ln y)\ \sin\frac{n\pi\theta}{\theta_1}\ .$$

3. The Complete Solutions

The complete solutions are obtained by superpositions of solutions (31), (34), and (40). From these complete solutions, the stresses and moments may be determined by use of the elastic law (2). Before obtaining the final results, it is necessary to determine the tangential and transverse shearing forces on the boundaries. These forces may be derived following Kirchhoff's theorem, see Reference (3).

Let S_s and T_s be the resultant shearing force per unit length in the transverse and tangential directions respectively due to shearing forces and twisting moments in a plane normal to the s-direction and S_θ and T_θ be those in a plane normal to θ-direction. Then it can be shown that

$$\begin{aligned} S_s &= Q_s + \frac{1}{s}\frac{M_{s\theta}}{\theta}\ \sec\alpha \\ T_s &= N_{s\theta} - \frac{M_{s\theta}}{S}\ \tan\alpha \\ S_\theta &= Q_\theta + \frac{\partial M_s}{\partial S} \\ T\ &= N_{\theta s} \end{aligned} \tag{41}$$

where Q_s and Q_θ are obtainable from the equilibrium equations (4).

Thus, by first combining solutions (31), (34) and (40) to obtain the complete solutions and then substituting these complete solutions into the elastic law (2) and equations (41) the resultant forces and moments may be determined. Note, however, that if the order of magnitude of the functions of y in U_{II}, V_{II} and W_{II} is the same as the order of k^o, then U_I, V_I, W_I, W_{II} and, the particular solutions, are of the same order of

k^0 while V_{II} is of $k^{1/4}$ and U_{II} is of $k^{1/2}$. (The assumption made will be justified later by a numerical example). Hence as k approaches zero asympototically,

$$U = U_I + U_p$$
$$V = V_I + V_p$$
$$W = W_I + W_{II} + W_p$$

Applying a similar argument to the resultant forces and moments yields the following results:

$$U = \frac{L}{E\delta}\frac{1}{\tan\alpha}\sum_n a_n\left\{\frac{C_{n1}}{m_n^2-1} + \frac{C_{n_1}}{m_n^2-2(1-\nu)}y^{-2} + \frac{C_{n_3}}{m_n^2}y^2 + \frac{m_n^2-4(1+\nu)C_{n_4}}{m_n^2[m_n^2-7+\nu]}y^{-4}\right.$$
$$\left. + [\tfrac{1}{3}m_n^2 - \tfrac{1}{2}(5-\nu) - \tfrac{1}{2}\frac{1+\nu}{m_n^2} + \tfrac{1}{3}(m_n^2-7+2\nu)\ \ln y]y^2\right\}\cos\frac{n\pi\theta}{\theta_1},$$

$$V = \frac{L}{E\delta}\frac{1}{\tan\alpha}\sum_n a_n\left\{\frac{C_{n^1}}{m_n^2-1} + \frac{2C_{n_2}}{m_n^2-2(1-\nu)}y-2 + \frac{3C_{n_4}}{m_n^2-7+2\nu}y-4\right.$$
$$\left. + \frac{1}{6}[3(1-2\nu) - m_n^2]y^2\right\}\sin\frac{n\pi\theta}{\theta_1},$$

$$W = \frac{L}{E\delta}\frac{1}{\tan^2\alpha}\sum_n a_n\left\{C_{n_1} + C_{n_2}y^{-2} + C_{n_3}y^2 + C_{n_4}y^{-4}\right.$$
$$+ y^{-1}[y^{\rho}(C_{n5}\cos(\rho\ln y) + C_{n6}\sin(\rho\ln y))$$
$$+ y^{-\rho}(C_{n7}\cos(\rho\ln y) + C_{n8}\sin(\rho\ln y))]$$
$$\left. + \frac{m_n^2}{3}[m_n^2 - 7 + 2\nu][1 + \ln y]\ y^2\right\}\sin\frac{n\pi\theta}{\theta_1},$$

$$\frac{\partial W}{\partial S} = \frac{1}{E\delta}\frac{1}{\tan^2\alpha}\frac{1}{2}\sum_n a_n y^{-3}\left\{y^{\rho}[(\rho-1)C_{n5} + \rho C_{n6})\cos(\rho\ln y)\right.$$
$$+((\rho-1)C_{n6} - \rho C_{n5})\sin(\rho\ln y)] + y^{-\rho}[(-(\rho+1)C_{n7} + \rho C_{n8})\cos(\rho\ln y)$$
$$\left. - ((\rho+1)C_{n8} + \rho C_{n7})\sin(\rho\ln y)]\right\}\sin\frac{n\pi\theta}{\theta_1},$$

$$N_s = \frac{L}{\tan\alpha}\sum_n a_n\left\{-2\left[\frac{C_{n_2}}{m_n^2-2(1-\nu)}y^{-2} + \frac{3C_{n_4}}{m_n^2-7+2}y^{-4}\right]\right. \qquad (42)$$
$$\left. + \frac{1}{6}[3-m_n^2]y^2\right\}\sin\frac{n\pi\theta}{\theta_1},$$

$$N_\theta = \frac{L}{\tan\alpha} \sum_n a_n y^2 \sin\frac{n\pi\theta}{\theta_1} ,$$

$$T_s = N_{s\theta} = T_\theta = \frac{L}{\tan\alpha} \sum_n a_n \left\{ \frac{6C_{n4}}{m_n(m_n^2-7+2\nu)} y^{-4} - \frac{m}{3} y^3 \right\} \cos\frac{n\pi\theta}{\theta_1} ,$$

$$M_s = \frac{L^2}{\tan^2\alpha} \frac{k\rho^2}{2(1-\nu^2)} \sum_n a_n y \left\{ y^\rho [C_{n6}\cos(\rho \ln y) - C_{n5}\sin(\rho \ln y)] \right.$$

$$\left. -y^{-\rho}[C_{n8}\cos(\rho \ln y) - C_{n7} \sin(\rho \ln y)] \right\} \sin\frac{n\pi\theta}{\theta_1} ,$$

$$M_\theta = \nu M_s ,$$

$$S_s = \frac{L}{\tan^2\alpha} \frac{k\rho^3}{4(1-\nu^2)} \sum_n a_n y^{-1} \left\{ y^\rho [(-C_{n5}+D_{n6}) \cos(\rho \ln y) \right.$$

$$- (C_{n6} + C_{n5}) \sin(\rho \ln y)] + y^{-\rho}[(C_{n7} + C_{n8}) \cos(\rho \ln y)$$

$$\left. + (C_{n8}-C_{n7}) \sin(\rho \ln y)] \right\} \sin\frac{n\pi\theta}{\theta_1} ,$$

$$S_\theta = \frac{L}{\tan^2\alpha} \frac{k\rho^2}{(1-\nu^2)} \sum_n a_n y^{-1} \left\{ y^\rho [(b_1 C_{n5} + b_2 C_{n6}) \cos(\rho \ln y) \right.$$

$$+ (b_1 C_{n6} - b_2 C_{n5}) \sin(\rho \ln y) + y^{-\rho} [(b_1 C_{n7} - b_2 C_{n8}) \cos(\rho \ln y)$$

$$\left. + (b_1 C_{n8} + b_2 C_{n7}) \sin(\rho \ln y)] \right\} \cos\frac{n\pi\theta}{\theta_1} ,$$

where $b_1 = \frac{3}{4}(1-\nu) \tan\alpha$, $b_2 = \frac{1}{2}(2-\nu)\, m_n$.

In the foregoing results, the two sets of homogeneous solutions are coupled only in the displacement W - the membrane forces are functions of the first set and particular solutions while the moments and transverse shearing forces are functions only of the second set of homogeneous solutions.

4. An Example - The Engine Shround

Consider the case of an engine shroud as an example. The smaller circular end of the engine shroud is fixed and the other end free.

Thus requires:

$$U = V = W = \frac{\partial W}{\partial S} = 0 \qquad \text{at } y \sqrt{\frac{L_1}{L}} = \xi \tag{43}$$

$$N_s = M_s = T_s = S_s = 0 \text{ at } y = 1. \tag{44}$$

The first four conditions are satisfied if, from the first four equations of equations (36),

$$\frac{1}{m_n^2-1} C_{n_1} + \frac{1}{m_n^2-2(1-\nu)} \frac{1}{\xi^2} C_{n_2} + \frac{\xi^2}{m_n^2} C_{n_3} + \frac{m_n^2-4(1+\nu)}{m_n^2[m_n^2-7+2\nu]} \xi^{-4} C_{n4}$$

$$+ \left[\frac{1}{3} m_n^2 - \frac{1}{2}(5-\nu) - \frac{1}{2}\frac{1+\nu}{m_n^2} + \frac{1}{3}(m_n^2-7+2\nu) \ln\xi\right] \xi^2 = 0 \tag{iv}$$

$$\frac{C_{n1}}{m_n^2-1} + \frac{2}{m_n^2-2(1-\nu)} \frac{1}{\xi^2} C_{n2} + \frac{3}{m_n^2-7+2\nu} \frac{1}{\xi^4} C_{n4} + \frac{1}{6}[3(1-2\nu)-m_n^2]\xi^2=0 \tag{iii}$$

$$C_{n1} + \frac{1}{\xi^2} C_{n2} + \xi^2 C_{n3} + \frac{1}{\xi^4} C_{n4} + \xi^{-1} \Big\{\xi^\rho[C_{n5} \cos(\rho \ln\xi)$$

$$+ C_{n6}\sin(\rho \ln\xi)] + \xi^\rho [C_{n7}\cos(\rho \ln\xi) + C_{n8}\sin(\rho \ln\xi)]\Big\}$$

$$+ \frac{m_n^2}{3} [m_n^2 - 7 + 2\nu][1 + \ln\xi]\xi^2 = 0 \tag{viii}$$

and

$$\xi^\rho\Big\{[(\rho-1)C_{n5} + C_{n6}] \cos(\rho \ln\xi) + [(\rho-1) C_{n6}-\rho C_{n5}] \sin(\rho \ln\xi)\Big\}$$

$$+\xi^{-\rho}\Big\{[-(\rho+1) C_{n7} + \rho C_{n8}] \cos(\rho \ln\xi) - [(\xi+1) C_{n8} + \rho C_{n7}] \sin(\rho \ln\ )\Big\} = 0 \tag{vii}$$

The other four conditions at y = 1 yield:

$$-2\left[\frac{1}{m_n^2-2(1-\nu)} C_{n2} + \frac{3}{m_n^2-7+2\nu} C_{n4}\right] + \frac{1}{6}[3-m_n^2] = 0 \tag{ii}$$

$$C_6 - C_8 = 0 \tag{v}$$

$$\frac{6}{m_n(m_n^2-7+2\nu)} C_{n4} - \frac{m_n}{3} = 0 \tag{i}$$

$$-C_{n5} + C_{n6} + C_{n7} + C_{n8} = 0 \tag{vi}$$

The eight constants in this particular case may be determined, one by one, by following the order of the equations indicated by Roman numerals. The results are as follows:

$$C_{n1} = \frac{m_n^2-1}{6}\left\{[m_n^2-3(1-2\nu)]\,\xi^2 + \frac{3(m_n^2-1)}{\xi^2} - \frac{m_n^2}{\xi^4}\right\}$$

$$C_{n2} = -\frac{1}{4}[m_n^2 - 2(1-\nu)]\,(m_n^2-1)$$

$$C_{n3} = -m_n^2\left\{\left[\frac{1}{3}m_n^2 - \frac{1}{2}(5-\nu) - \frac{1}{2}\frac{1+\nu}{m_n^2}\right] + \frac{1}{3}(m_n^2-7+2\nu)\ln\xi + \frac{1}{6}[m_n^2 - 3(1-2\nu)] + \frac{1}{4}\frac{m_n^2-1}{\xi^4} + \frac{1}{9}\frac{m_n^2 + 2(1+\nu)}{\xi^6}\right\}$$

$$C_{n4} = \frac{m_n^2[m_n^2 - 7 + 2\nu]}{18}$$

$$C_{n5} = (2+\beta)\,C_{n6} \tag{45}$$

where

$$\beta = \frac{\xi^{\rho}[(3\rho-2)\cos(\rho\ln\xi)-(\rho+1)\sin(\rho\ln\xi)] + \xi^{-\rho}[\rho\cos(\rho\ln\xi)-(\rho+1)\sin(\rho\ln\xi)]}{\xi^{\rho}[\rho\sin(\rho\ln\xi)-(\rho-1)\cos(\rho\ln\xi)] + \xi^{-\rho}[\rho\sin(\rho\ln\xi)+(\rho+1)\cos(\rho\ln\xi)]}$$

$$C_6 = \frac{\frac{m_n^2}{3}[(m_n^2-7+2\nu)(1+\ln\xi)]\xi^3 + C_1\xi + C_2\xi^{-1} + C_3\xi^3 + C_4\xi^{-3}}{\xi^{\rho}[(2+\beta)\cos(\rho\ln\xi) + \sin(\rho\ln\xi)] + \xi^{-\rho}[\beta\cos(\rho\ln\xi) + \sin(\rho\ln\xi)]}$$

$$C_7 = \beta C_6$$

and

$$C_8 = C_6$$

The following data are used in the numerical computations:

$$\theta_1 = 180^\circ, \quad \alpha = 75^\circ, \quad \nu = \frac{1}{3}, \quad L = 373.312'' \tag{46}$$

Two types of lateral normal load are considered. One type is symmetrical, the other, antisymmetrical. The symmetrical load is assumed to be of form

$$p = p\sin\theta \qquad \leq \theta \leq \pi \tag{47}$$

Hence refering to equation (9)

$$a_n = p \qquad \text{for } n = 1$$

$$\;\;\; = 0 \qquad \text{for } n > 1 \tag{48}$$

The antisymmetric load is assumed to be of the form

$$p = \frac{4}{9}\sqrt{3}\ p\,(\sin\theta + \frac{1}{2}\sin 2\theta), \qquad (49)$$

Thus

$$\begin{aligned} a_n &= \frac{4}{9}\sqrt{3}\ p & \text{for } n = 1 \\ &= \frac{1}{2}\left(\frac{4}{9}\sqrt{3}\ p\right) & n = 2 \\ &= 0 & n > 0 \end{aligned} \qquad (50)$$

The two types of loading are depicted in Fig. (2).

Thus, for the cases mentioned above, only the solutions for n = 1 and 2 need to be carried out. The only parameter still to be determined is the thickness t at s = L denoted by t'. The thickness is determined by a trial-and-error method. To cover a wide range of the possible thickness, t' = 0.4", 0.6" and 0.8" are assumed. Refering to equations (1) and (6a)[+] and (29) the corresponding values of k are computed, and given in Table 1.

TABLE I. THE VALUES OF k AND ρ

t'	0.4"	0.6"	0.8"
k	8.927×10^{-8}	2.008×10^{-7}	3.571×10^{-7}
ρ	153.48	125.32	108.53

Substituting the known quantities into expressions (45) the eight constants are readily computed. In turn, the displacements and forces may be obtained from equations (42). The computations have been programmed in Fortran Language and carried out by the University's Univac Solid State 80 Computer. The results are presented in the following form

[+]For stiffened shells, see p.p. 22 and 23 Reference (1). For such cases, instead of the thickness, the values of the parameter k are concerned.

$$F(y,\theta) = Cf_n(y) \begin{Bmatrix} \sin \dfrac{n\pi\theta}{\theta_1} \\ \cos \dfrac{n\pi\theta}{\theta_1} \end{Bmatrix} \tag{51}$$

where C is a coefficient; $C = \frac{L^2}{Et'} p$ for displacements, $C = L^2 p$ for moments, $C = Lp$ for forces and $f_n(y)$ are dimensionless functions, whose numerical values are tabulated in Tables III to V, and depicted in Figures 3 to 14, for $\frac{L_1}{L}=0.81$ or $\xi = 0.90$.

The assumption made in determining the order of magnitude of the solutions that the functions of y are of the order of k^o, now may be justified by the following facts for the present case. The constants C_{n5}, C_{n7}, and C_{n8}, obtained from equations (39) are of the same order as C_{n6} which is of the order of $\xi^{-\rho}$. Thus, the y-functions are of the order of $(\frac{y}{\xi})^{\rho}$ or k^o. Because the bending effects are confined to a narrow region near the fixed edge where $y = \xi$.

In this example, the deflection of the free end given by Figures (5) and (11), is comparatively large to the thickness. For such large displacement, the theory is applicable provided that the shell is not overstrained [6]. Therefore the strains at the fixed end control the validity of the results.

IV. CLOSING REMARKS

The key to the present solution lies in the development of a new method of solving the algebric equation. The method is quite similar to the so called perturbation method. The perturbation method, however, is used in expanding differential equations.

This method is likely good for the present problem only. The lateral deflection of cylinderical shells, for example, is governed by the following equation, from Reference (4).

$$k \nabla^8 w \; t(1-\nu)^2 \; w^{IV} = 0$$

or

$$kf(\lambda) + (1-\nu)^2\lambda^4 = 0$$

Let

$$\lambda^2 = x_0 + kx_1,$$

then

$$k[f(x_0) + 2x_1x_0(1-\nu^2)] + (1-\nu)x_0^2 = 0$$

Therefore, $x_0 = 0$ but x_1 is not determined. Thus, this approach fails although $X_0^2 = 0$ does provide the roots of membrane theory.

The solution obtained is an approximate one. Its accuracy depends on the parameters k and $m = (\frac{m\pi}{\theta_1} \sec \alpha)$. Better results will be obtained for smaller values of k and m. An estimate of the error involved in the example in using the asymptotic roots of λ is made by comparison of the asymptotic values with those computed from expressions (25) and (26). The comparison is shown in Table 2, for $\theta_1 = \pi$ and $\alpha = 75^{\circ}$.

TABLE II. THE VALUES OF λ

λ	t'	n = 1	n = 2	Asymptotic Values
$\lambda_{1,2}$	0.4''	±0.999999	±1.0523	±1
	0.6''	±0.999997	±1.1142	±1
	0.8''	±0.999995	±1.1955	±1
$\lambda_{3,4}$	0.4''	±3.00003	±2.9851	±3
	0.6''	±3.00007	±2.9663	±3
	0.8''	±3.00013	±2.9397	±3
$\lambda_{5,6,7,8}$	0.4''	±153.27(1.0027±i)	±152.75(1.0099±i)	±153.48(1±i)
	0.6''	±125.09(1.0035±i)	±124.51(1.0149±i)	±125.32(1±i)
	0.8''	±108.28(1.0045±i)	±107.77(1.0198±i)	±108.53(1±i)

It is clear that when n = 1, the solution will result in very good accuracy. When n = 2, some error is introduced. However, the load amplitude of the function for n = 2 is only one half of that for the function of n = 1 in the case of the antisymmetrical load. Consequently the error will be proportionally reduced. Thus, one may conclude that the results presented in Figs. 3 to 14 would be good enough for practical design purposes. Should a better error estimate be desirable, it is necessary to elaborate the solutions based on the roots computed from expressions (25) and (26). Nevertheless, the present solution is an exact asymptotic solution.

This asymptotic solution helps in understanding how the displacements and stresses function in a cone structure. The membrane displacement and stresses predominate most of the region of the cone. The bending effect is pertinent only near the supporting edges. This effect, called edge effect or boundary layer phenomenon, has been observed in spherical shells, in Reference [5]. The lateral displacement w functions as a bridge between the membrane and bending effects.

The present results could be converted into solutions for a complete cone by interchanging the sinusoidal functions in the displacements. As a matter of fact, the eight roots of λ remain in the same.

V. ACKNOWLEDGMENT

The author wishes to thank Professor W. K. Rey, Director of the project, for his reading over of the manuscript.

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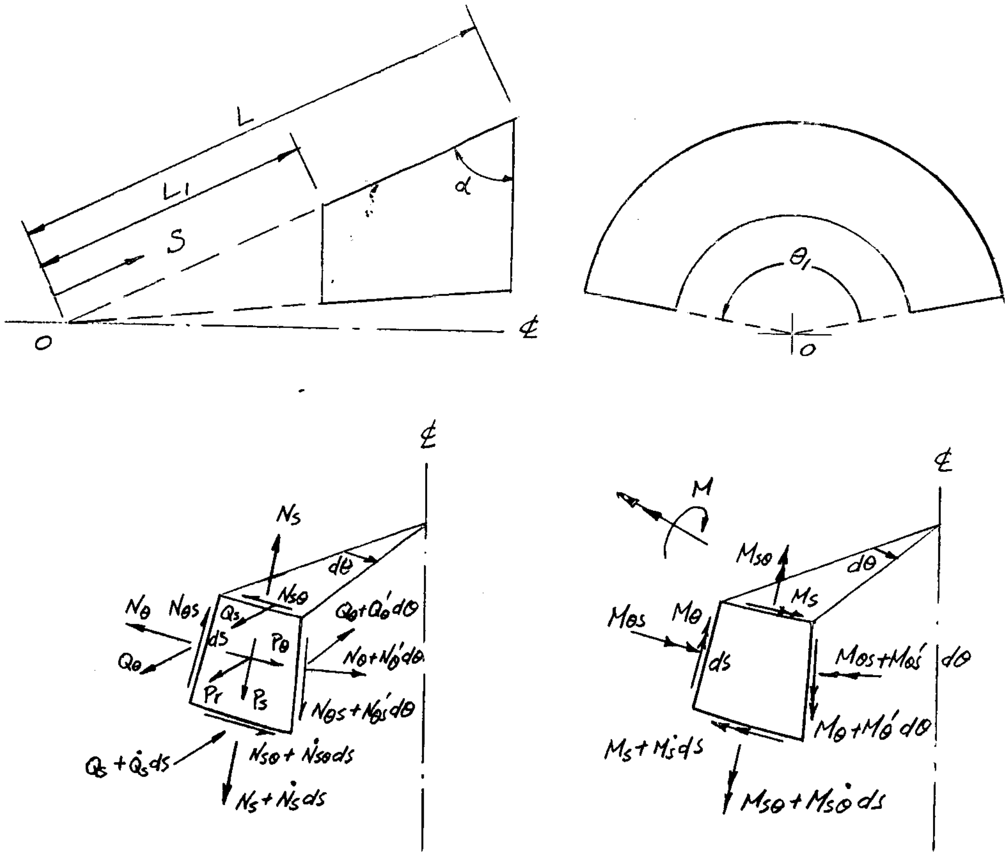
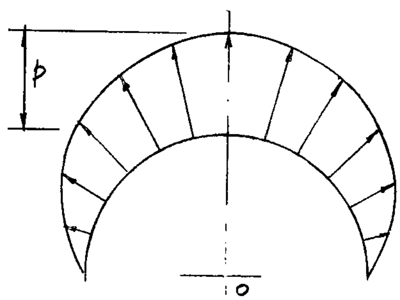
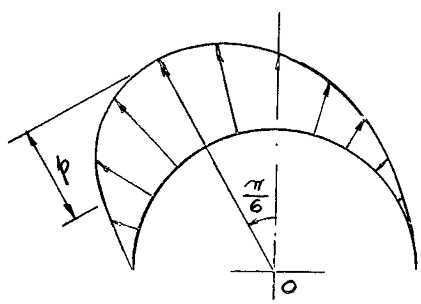


Fig. 1



The Symmetrical Load



The Asymmetrical Load

Fig. 2

TABLE III MEMBRANE DISPLACEMENTS AND FORCES

Y	n = 1						n = 2					
$= \frac{S}{L}$	$U \frac{Et'}{L^2 p}$	$V \frac{Et'}{L^2 p}$	$N_s \frac{1}{Lp}$	$N \frac{1}{Lp}$	$T_s \frac{1}{Lp}$	$W \frac{Et'}{Lp}$	$U \frac{Et'}{L^2 p}$	$V \frac{Et'}{L^2 p}$	$N_s \frac{1}{Lp}$	$N \frac{1}{Lp}$	$T_s \frac{1}{Lp}$	$W \frac{Et'}{Lp}$
1.00	.092	-.030	0	.268	0	2.43	.212	-.0508	0	.268	0	7.28
0.99	.087	-.028	-.006	.263	.021	2.35	.199	-.0489	-.008	.263	.042	6.89
0.98	.082	-.026	-.014	.257	.043	2.25	.184	-.0468	-.024	.257	.085	6.46
0.97	.075	-.024	-.024	.252	.065	2.14	.167	-.0444	-.047	.252	.130	5.98
0.96	.068	-.022	-.036	.247	.088	2.02	.148	-.0414	-.078	.247	.176	5.45
0.95	.059	-.019	-.050	.242	.112	1.89	.128	-.0376	-.118	.242	.224	4.87
0.94	.049	-.016	-.066	.237	.137	1.74	.106	-.0329	-.166	.237	.274	4.27
0.93	.039	-.013	-.085	.232	.163	1.58	.082	-.0271	-.225	.232	.325	3.65
0.92	.027	-.010	-.107	.227	.189	1.44	.056	-.0198	-.294	.227	.379	2.90
0.91	.014	-.005	-.132	.222	.217	1.05	.029	-.0109	-.375	.222	.435	1.62
0.90	0	0	-.160	.217	.246	0	0	0	-.468	.217	.492	0

TABLE IV TRANSVERSE SHEARING FORCES AND NORMAL MOMENT $n = 1$

Y	t' = 0.4" (k = 8.927×10^{-8})				t' = 0.6" (k = 2.008×10^{-7})				t' = 0.8" (k = 3.571×10^{-7})			
$= \frac{S}{L}$	W $\frac{Et'}{L^2 p}$	S $\frac{1}{Lp} \times 10^3$	S_s $\frac{1}{Lp} \times 10^3$	M_s $\frac{1}{L^2 p} \times 10^5$	W $\frac{Et'}{L^2 p}$	S $\frac{1}{Lp} \times 10^3$	S_s $\frac{1}{Lp} \times 10^3$	M_s $\frac{1}{L^2 p} \times 10^5$	W $\frac{Et'}{L^2 p}$	S $\frac{1}{Lp} \times 10^3$	S_s $\frac{1}{Lp} \times 10^3$	M_s $\frac{1}{L^2 p} \times 10^5$
.95	1.89	.002	0.00	0.00	1.88	0.000	-0.01	0.00	1.88	.003	- .05	.05
.94	1.74	-.001	- .01	0.01	1.74	.005	-0.04	0.07	1.74	.016	0.00	+ .13
.93	1.58	.005	- .03	+0.06	1.60	.014	0.15	+0.04	1.61	+.008	0.47	- .21
.928	1.55	.007	0.00	+0.06	1.57	+.012	0.26	-0.03	1.58	-.005	0.64	- .42
.924	1.50	+.009	0.14	+0.02	1.51	-.005	0.58	-0.33	1.51	-.053	1.00	-1.02
.920	1.44	-.001	0.44	-0.18	1.44	-.051	0.93	-0.89	1.40	-.139	1.21	-1.85
.916	1.36	-.038	0.78	-0.63	1.30	-.130	1.03	-1.63	1.24	-.262	0.90	-2.67
.912	1.19	-.114	0.75	-1.24	1.08	-.251	+0.26	-2.18	0.98	-.391	- 0.60	-2.96
.910	1.05	-.162	+ 0.29	-1.44	0.92	-.302	-0.76	-2.10	0.82	-.434	- 2.04	-2.39
.908	0.87	-.205	- 0.72	-1.37	0.73	-.332	-2.37	-1.54	0.64	-.440	- 4.08	-1.29
.904	0.38	-.185	- 5.18	0.54	0.30	-.223	-7.86	1.97	0.25	-.240	-10.21	3.72
.900	0.00	0.234	- 13.20	6.98	0.00	0.353	-16.20	10.49	0	0.473	-18.67	14.00

TABLE V TRANSVERSE SHEARING FORCES AND NORMAL MOMENT $n = 2$

Y =	t' = 0.4" ($k = 8.927 \times 10^{-8}$)				t' = 0.6" ($k = 2.008 \times 10^{-7}$)				t' = 0.8" ($k = 3.571 \times 10^{-7}$)			
	W $\frac{Et'}{L^2 p}$	S $\frac{1}{Lp} \times 10^3$	S_s $\frac{1}{Lp} \times 10^5$	M_s $\frac{1}{L^2 p} \times 10^5$	W $\frac{Et'}{L^2 p}$	S $\frac{1}{Lp} \times 10^3$	S_s $\frac{1}{Lp} \times 10^5$	M_s $\frac{1}{L^2 p} \times 10^5$	W $\frac{Et'}{L^2 p}$	S $\frac{1}{Lp} \times 10^3$	S_s $\frac{1}{Lp} \times 10$	M_s $\frac{1}{L^2 p} \times 10^5$
.95	4.87				4.87	.00	-0.02	0.01	4.87	.01	- 0.02	
.94	4.25	0.00			4.26	.01	-0.06	0.09	4.27	+.03	- 0.06	+0.02
.93	3.60	0.01			3.63	+.02	0.20	+0.06	3.65	-0.01	0.00	-0.03
.928	3.47	0.02	.00	+.009	3.50	+.01	0.36	-0.41	3.51	-0.05	0.09	-0.06
.924	3.21	+0.01	.20	+.003	3.24	-.04	0.79	-0.46	3.23	-0.18	0.14	-0.14
.920	2.95	-0.02	.59	-.025	2.94	-0.16	1.28	-1.23	2.90	-0.39	0.17	-0.26
.916	2.65	-0.20	1.08	-.087	2.57	-0.36	1.43	-2.25	2.48	-0.64	+ 0.01	-0.37
.912	2.23	-0.29	1.04	-1.71	2.07	-0.58	0.36	-3.00	1.94	-0.84	- 0.08	-0.39
.910	1.94	-0.38	+ .40	-1.98	1.75	-0.64	-1.04	-2.90	1.62	-0.86	- 0.28	-0.33
.908	1.58	-0.43	- .99	-1.90	1.39	-0.62	-3.28	-2.13	1.27	-0.75	- 0.56	-1.78
.904	0.71	-0.20	- 7.15	0.74	0.60	-0.09	-10.85	2.72	0.54	0.07	- 1.41	5.13
.900	0	1.09	-18.20	9.64	0	+1.64	-22.21	14.47	0	2.19	-25.77	19.32

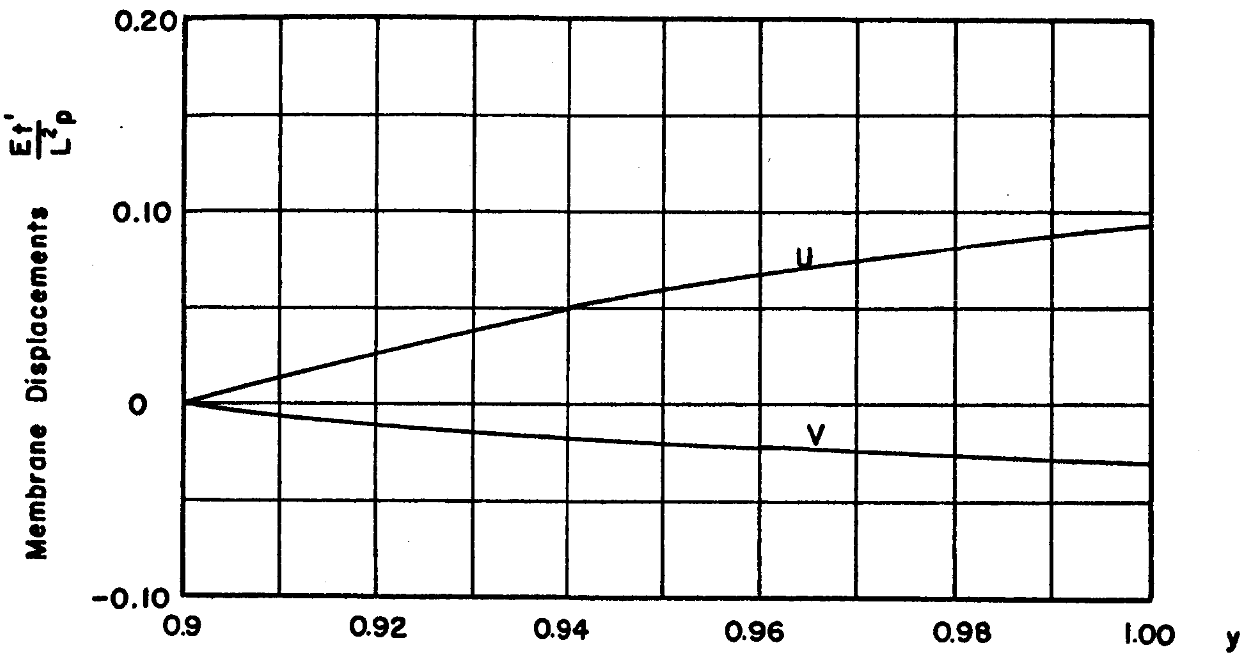


Fig. 3 Membrane Displacements U and V. (n = 1)

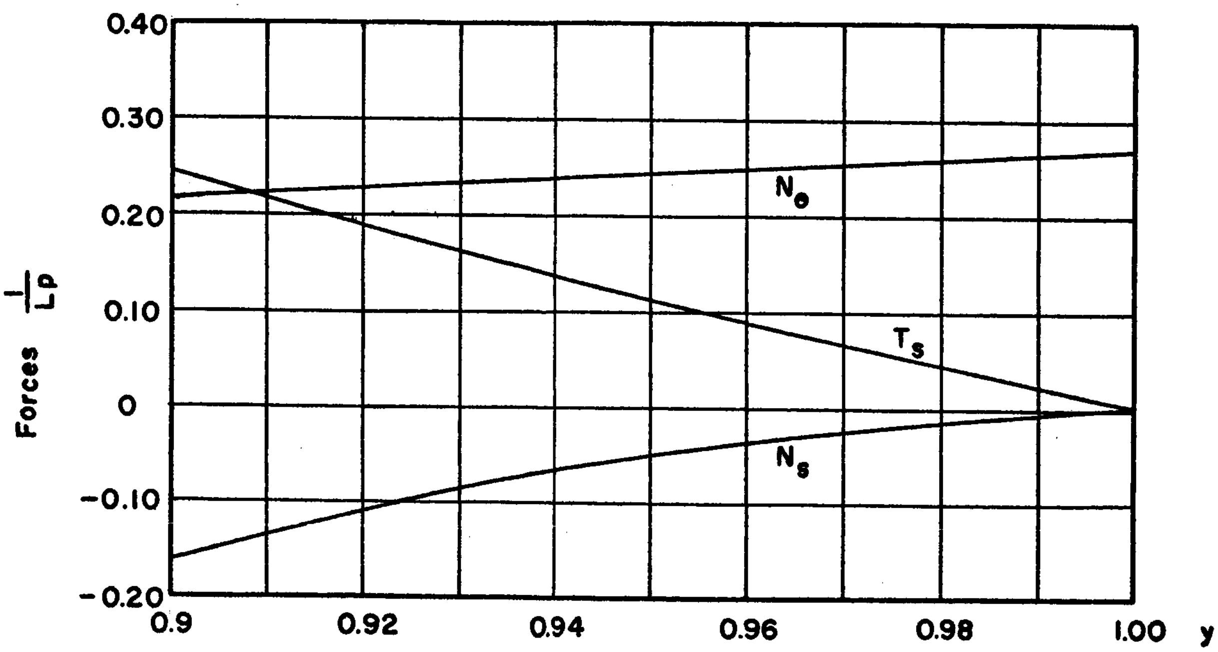


Fig. 4 Membrane Forces N_θ, T_s, and N_s (n = 1)

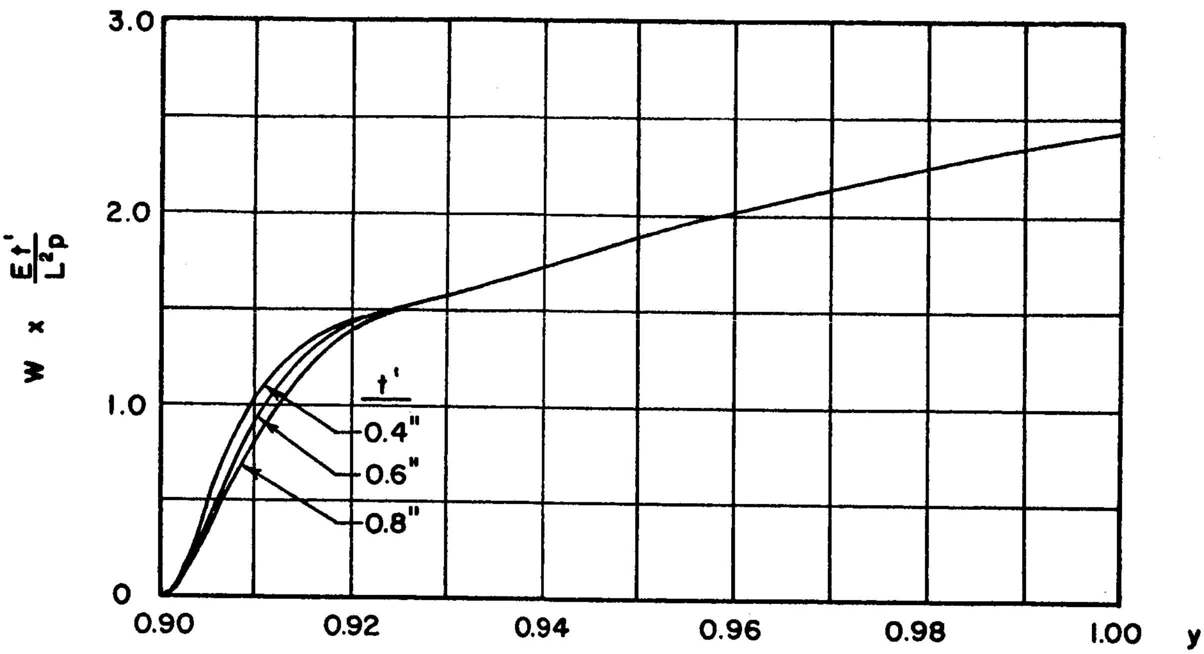


Fig. 5 Normal Displacement W (n=1)

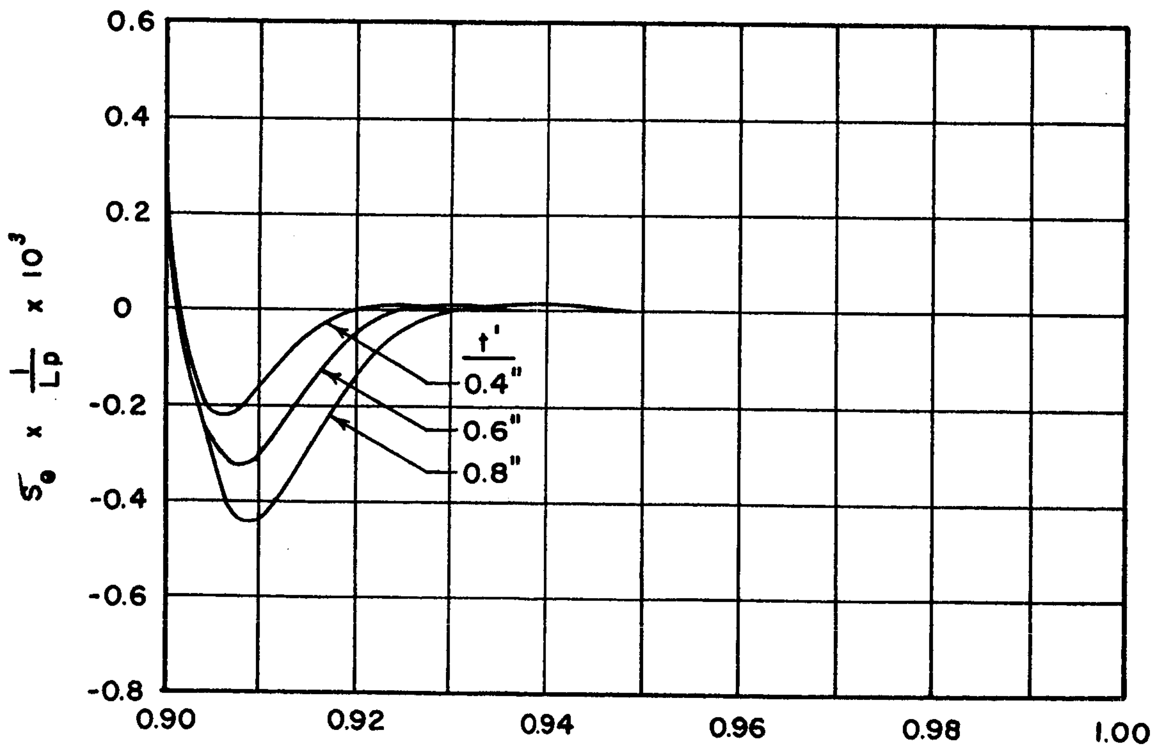


Fig. 6 Transverse Shearing Force S_θ (n=1)

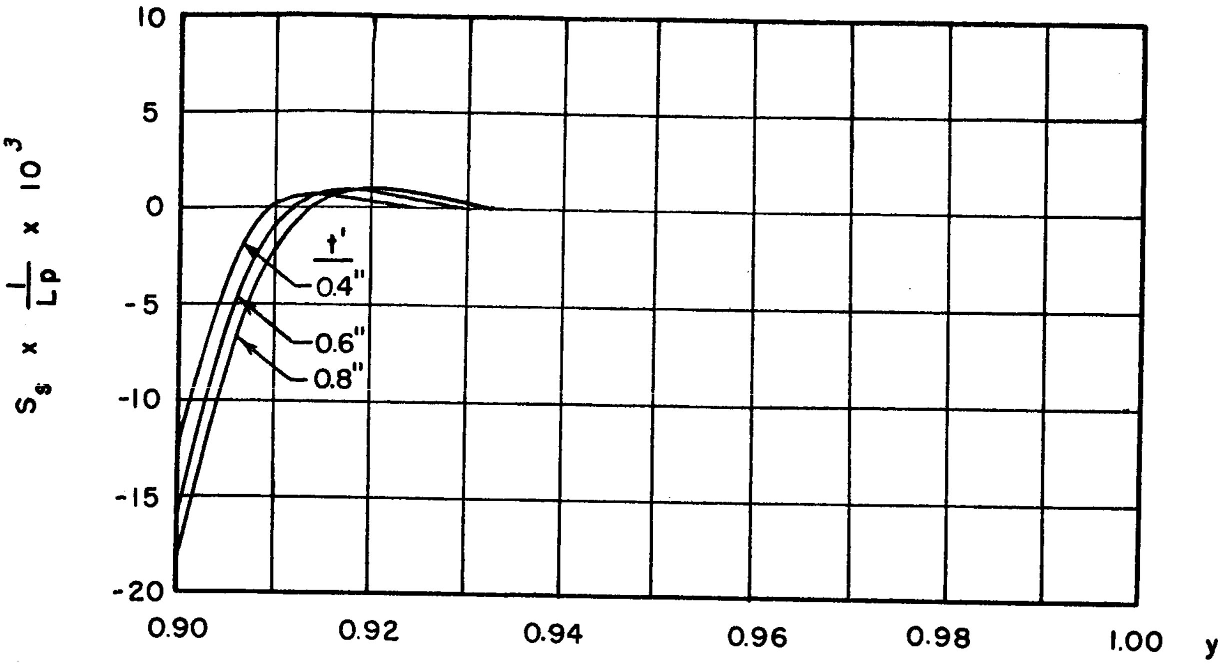


Fig. 7 Transverse Shearing Force S_s (n = 1)

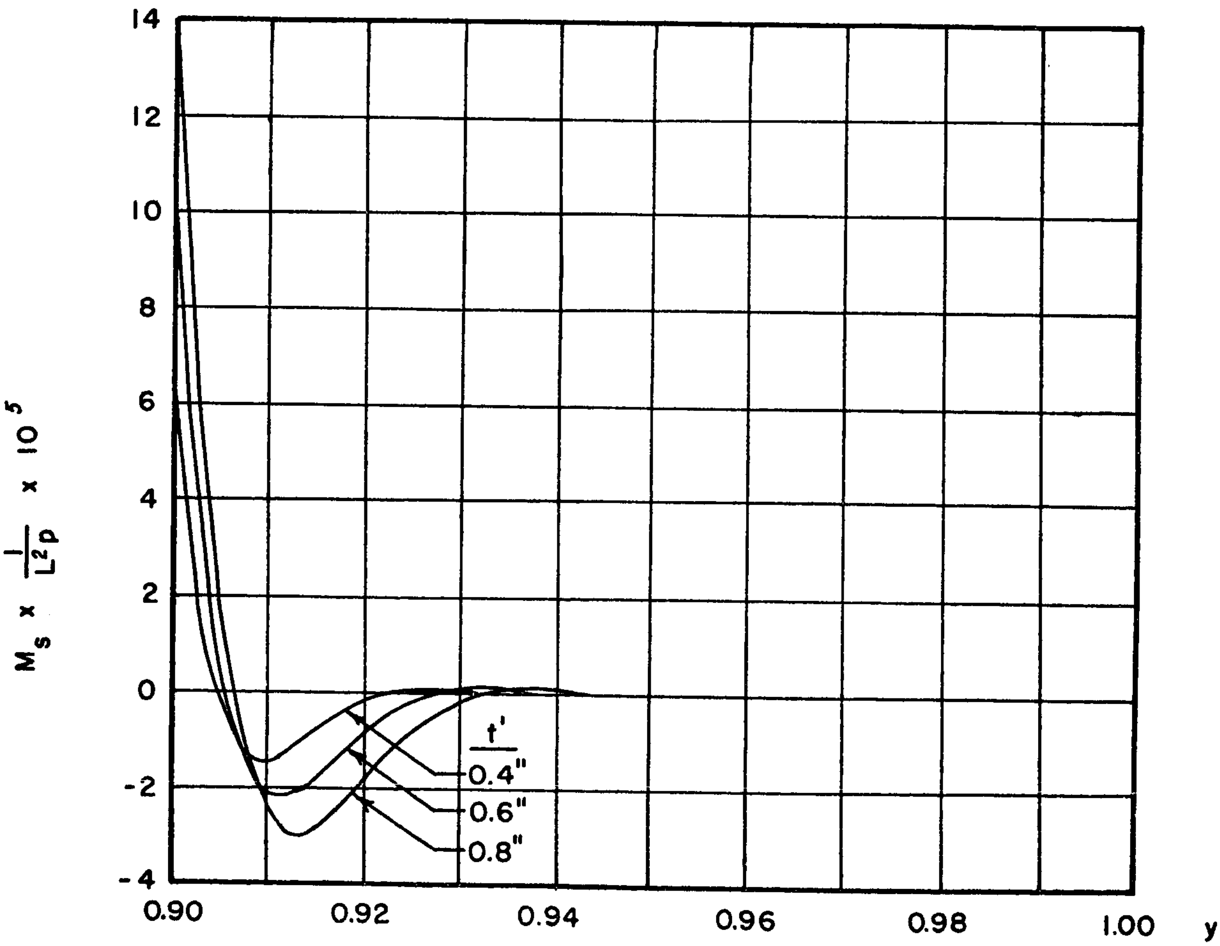


Fig. 8 Normal Moment M_s (n = 1)

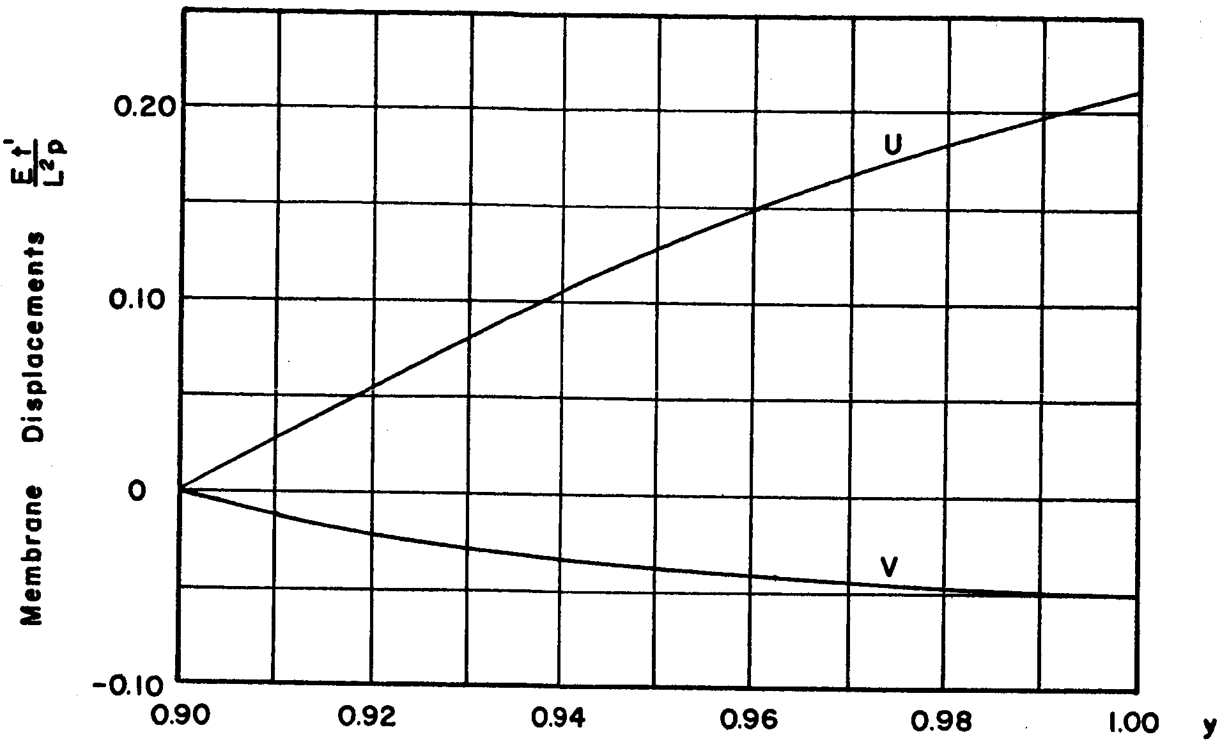


Fig. 9 Membrane Displacements U and V (n=2)

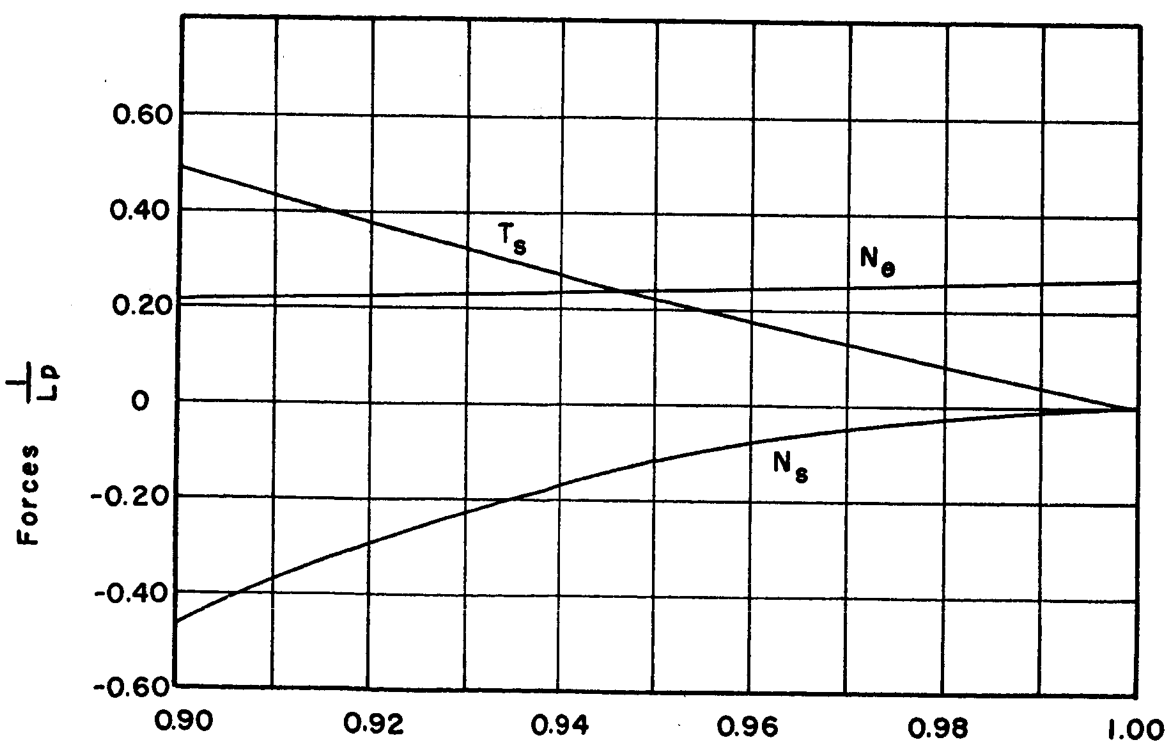


Fig. 10 Membrane Forces N_θ, T_s, and N_s (n=2)

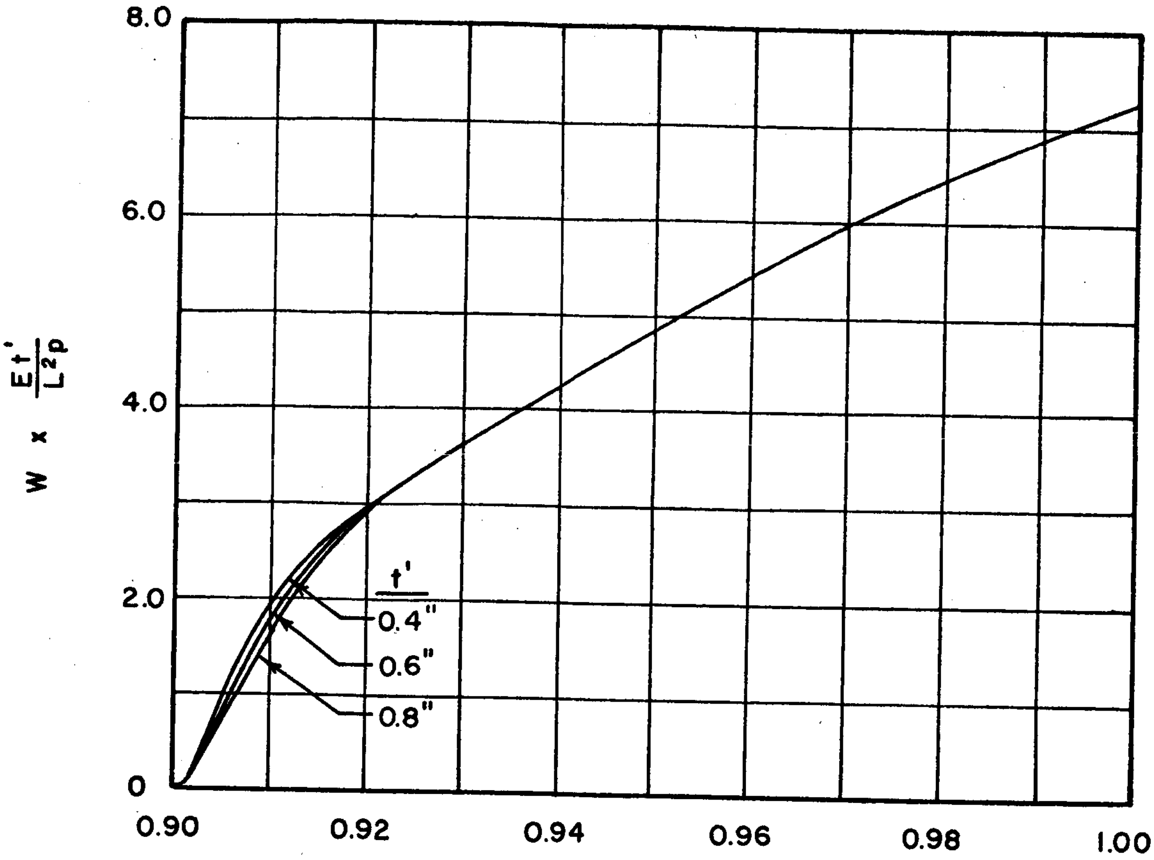


Fig. 11 Normal Displacement W (n = 2)

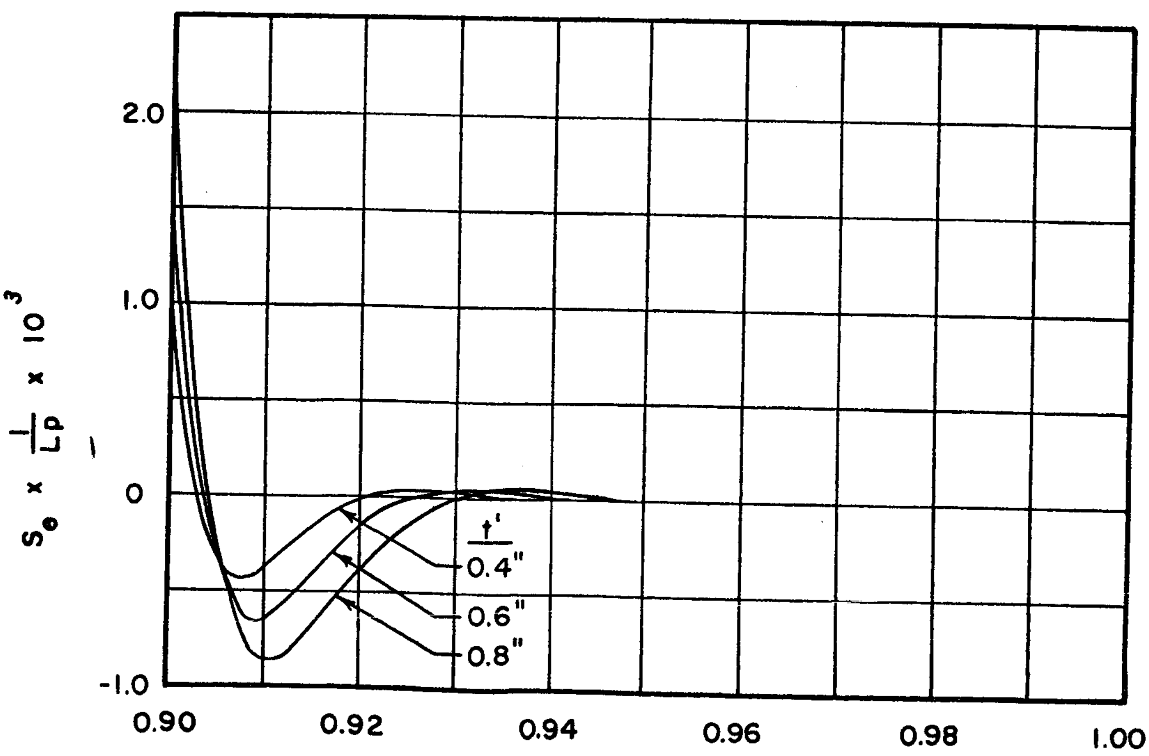


Fig. 12 Transverse Shearing Force S_θ (n = 2)

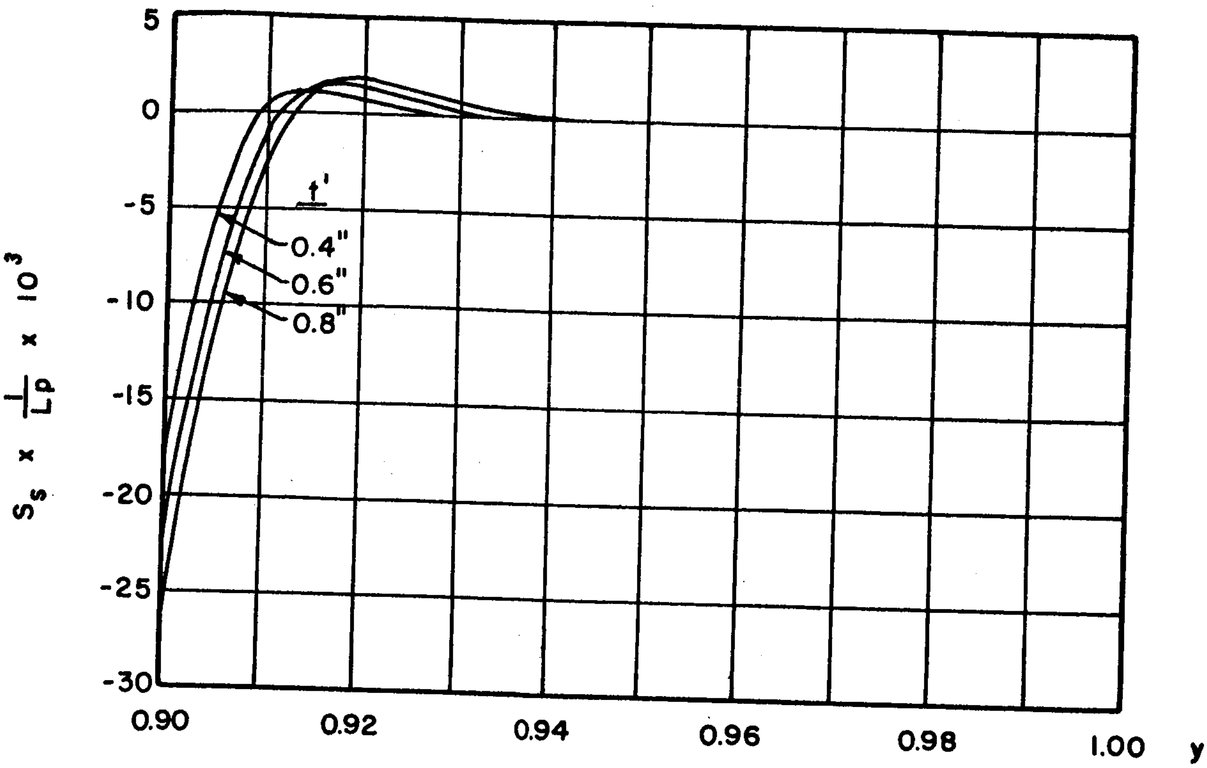


Fig. 13 Transverse Shearing Force S_s (n = 2)

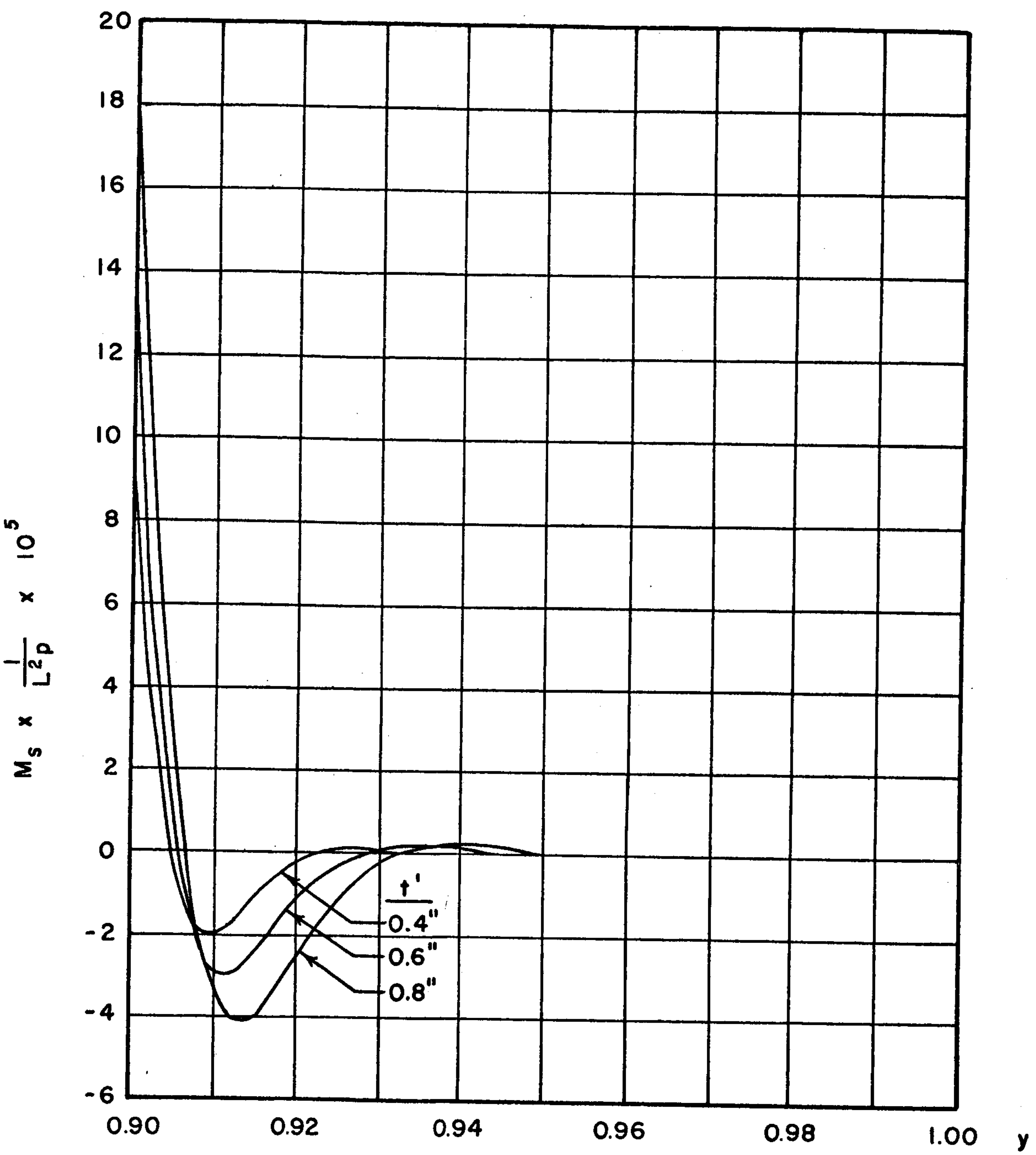


Fig. 14 Normal Moment M_s (n = 2)

APPENDIX I. ON THE APPLICATION OF THE RESULTS

1. Take the case when $t' = 0.6''$ or $k = 2.008 \times 10^{-7}$ as an example for the purpose of illustration. In what follows, only the displacement of the free end and the forces at the fixed end will be concerned.

A. When the lateral load is symmetrical, the maximum values of the normal displacement and forces are easily obtained by referring to Figures 3 to 8 and Tables III and IV. For example, taking $E = 10 \times 10^6$ lb. per square inch, one has the following maximum values:

$$W = 2.433 \times \frac{Lp}{E} = 0.0564\ p \text{ in.} \qquad \text{at } \theta = \frac{\pi}{2},$$

$$M_s = 1.049 \times 10^{-4}\ L^2\ p = 14.6\ p\ \frac{\text{in-lb}}{\text{in}} \qquad \text{at } \theta = \frac{\pi}{2},$$

$$S = 3.535 \times 10^{-4}\ L\ p = 0.1320\ p\ \frac{\text{lb}}{\text{in}} \qquad \text{at } \theta = 0,\ \pi,$$

$$T_s = 0.2462 \times L\ p = 91.7\ p\ \frac{\text{lb}}{\text{in}} \qquad \text{at } \theta = 0,\ \pi,$$

$$N_s = -0.1598 \times L\ p = -59.6\ p\ \frac{\text{lb}}{\text{in}} \qquad \text{at } \theta = \frac{\pi}{2}$$

where p is the maximum load intensity in lb. per square inch. It is negative when it acts toward the center of the cone.

B. When the load is antisymmetrical, the maximum displacements and forces occur at angles which make the first derivative with respect to θ vanish. For example,

$$M_s = \frac{4}{9}\sqrt{3}\ [1.049 \sin\theta + \frac{1}{2}\ 1.447 \sin 2\theta\ ] \times 10^{-4}\ L^2\ p$$

Let

$$\frac{dM_s}{d\theta} = 0$$

then

$$\theta = 57^\circ 30'$$

and

$$M_s\big]_{max} = 16.41\ \frac{\text{in-lb}}{\text{in.}}$$

In a similar way,

$$N_s = -\frac{4}{9}\sqrt{3}\,[\,0.1598 \sin\theta + \frac{1}{2}(0.4684)\sin 2\theta\,]\ L\ p$$

$$N_s]_{max} = -101.2\ p\ \frac{lb}{in} \qquad \text{at } \theta = 51^{\circ}00'$$

$$W = \frac{4}{9}\sqrt{3}\,[\,2.433 \sin\theta + \frac{1}{2}(7.282)\sin 2\theta\,]\ \frac{L^2}{Et'}\ p$$

$$W]_{max} = 0.0974\ p\ in \qquad \text{at } \theta = 51^{\circ}00'$$

The following forces are maximum at $\theta = 0$.

$$S_\theta = \frac{4}{9}\sqrt{3}\,[\,3.535 \cos\theta + \frac{1}{2}(16.38)\cos 2\theta\,]\ 10^{-4}\ L\ p$$

$$S_\theta] = 0.3370\ p\ \frac{lb}{in}$$

$$T_s = \frac{4}{9}\sqrt{3}\,[\,0.2462 \cos\theta + \frac{1}{2}(0.4924)\cos 2\theta\,]\ L\ p$$

$$T_s]_{max} = 141.\ p\ \frac{lb}{in}$$

Note that there are considerable differences between the results of the symmetrical and antisymmetrical loadings at the same load intensity.

2. The order of magnitude of the transverse shearing forces, S_θ, in both the cases of symmetrical and antisymmetrical loadings is the lowest among the forces. Further, the small shearing forces carried by supports along the two straight edges are appreciable only near the fixed end of the segment, as is shown by the curves for S_θ in Figures 6 and 12. This fact justifies the alteration of the free straight edges of the engine shroud to edges which are free from normal moments and forces but not transverse shearing forces. However, the computed shearing forces should be taken into consideration in the design of the side edges.

APPENDIX II. NOTATION

The following symbols have been adopted for use in this report:

a_n = coefficient of lateral normal loads defined in Eq. (9);

A_n, B_n, C_n = coefficients of solutions of displacements defined in Eqs. (8);

d_{ij} = coefficients given by Eqs. (13);

D = tensional rigidity of shell = $\frac{Et}{1-\nu^2}$;

E = Young's modulus of elasticity;

G, g_n = coefficients defined in Eqs. (18);

k = parameter of bending effect = $\frac{\delta^2}{12}$;

K = bending rigidity of shell = $\frac{Et^3}{12(1-\nu^2)}$;

L, L_1 = distances measured from the apex along the conical surface to the free end and fixed end of the segment of the cone respectively;

m_n, m = $\frac{n\pi}{\theta_1}$ sec α ;

M_s, M_θ = normal moments per unit length in planes perpendicular to s and θ directions;

$M_{s\theta}$, $M_{\theta s}$ = twisting moments per unit length in planes perpendicular to s and θ directions respectively;

N_s, N_θ = normal forces per unit length in planes perpendicular to s and θ directions respectively;

$N_{s\theta}$, $N_{\theta s}$ = tangential shearing-forces per unit length in planes perpendicular to s and θ directions respectively;

P_s, P_θ, P_r = surface loads per unit area in the directions of s, θ and the normal-to-the-middle-surface respectively;

Q_s, Q_θ = transverse shearing forces per unit length in planes perpendicular to s and θ directions respectively;

s	= distance measured from the apex along the conical surface;
S_s, S_θ	= resultant transverse shearing forces due to Q_s, Q_θ, $M_{s\theta}$ and $M_{\theta s}$;
t	= thickness of shell;
t'	= thickness of shell at $s = L$;
T_s, T_θ	= resultant tangential shearing forces due to $N_{s\theta}$, $N_{\theta s}$, $M_{s\theta}$ and $M_{\theta s}$;
U, V, W	= components of displacement in the directions of s, θ and the normal-to-the-middle-surface directions respectively;
U_1, V_1, W_1	= first set of homogeneous solutions of U, V, and W;
U_{11}, V_{11}, W_{11}	= second set of homogeneous solutions of U, V, and W;
U_p, V_p, W_p	= particular solutions of U, V, and W;
x_0, x_1	= unknowns defined in Eq. (19);
y	$= \sqrt{\frac{s}{L}}$;
α	= angle between the conical surface and a plane perpendicular to the axis of the cone;
γ	= constant defined in Eq. (25);
δ	$= \frac{t}{s}$;
θ	= angle between two meridians;
θ_1	= angle between two edge-meridians of the shell segment;
λ_n	= characteristic constant defined in Eq. (11);
λ_i, λ_j	= eight roots of λ, $i = 1,2,3$ and 4, $j = 5,6,7$ and 8.
ν	= Poisson's ratio;

$$\rho = \left| \frac{\sqrt{2}}{2} \left(\frac{G}{K} \right)^{1/4} \right|$$

Differential Notation: Differentiation with respect to s and θ coordinates are indicated by dot (·) and prime (,) respectively.

APPENDIX D

LITERATURE SURVEY WITH ABSTRACTS

Prepared by

Raymond C. Montgomery

This appendix was previously submitted as Technical Report D for contract NAS 8-5168.

Technical Report D for NASA Contract NAS8-5168

Literature Survey With Abstracts

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Prepared By
Raymond C. Montgomery

During the performance of NASA Contract NAS8-5168, a literature survey was undertaken to identify current publications pertaining to the subject matter included in the work scope of this contract. Through the facilities of the Office of Scientific Information of NASA and the Defense Documentation Center for Scientific and Technical Information, copies were obtained of many of the publications that appeared to contain information of value to the members of the project staff. Abstracts were prepared of those publications that contained particularly useful data or information.

This report is presented in two parts. Part I consists of a list of all the publications that have been obtained and are now available for use in the Library of the Department of Aerospace Engineering. Part II consists of the abstracts that were prepared during the performance of Contract NAS8-5168.

Part I - List of Publications

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280. Yoshimura, Y. and Niisawa, J.: Lower Buckling Stress of Circular Cylindrical Shells Subjected to Torsion. Journal of Aeronautical Science, Vol. 24, 1957, pp. 211-216.

281. Yu, Y. Y.: Nonlinear Flexural Vibrations of Sandwich Plates. Polytechnic Institute of Brooklyn, Air Force Office of Scientific Research Contract, AFOSR 1310, February 1962.

282. Yu, Y. Y.: Application of Variational Equation of Motion to the Nonlinear Vibration Analysis of Homogeneous and Layered Plates and Shells. Air Force Office of Scientific Research, AFOSR 2256, February 1962.

283. Yuan, S. W.: Thin Cylindrical Shells Subjected to Concentrated Loads. Quarterly of Applied Mathematics, Vol. 4, No. 1, 1946, pp. 13-26.

Part II - Abstracts

1. Amiro, I. Y.: Investigating the Stability of a Ribbed Cylindrical Shell During Longitudinal Compression. Foreign Technology Division, Air Force Systems Command, FTD-TT-62-1622. (Translated from: Ukrainian Periodical, Prykladna Mekhanika, Vol.6, Nr.3, 1960, pp. 272-280.)

Critical buckling stress in stiffened circular cylinders with discrete stiffeners with rigid closed ends is considered for the case of longitudinal compression.

A form of the buckled mode shape is used that is consistent with the predictions of thin shell theory (without discrete stiffeners). The assumed mode shape is introduced into the compatability relation yielding a stress function. The internal energy is expressed as the sum of the energy stored in the shell, the stiffeners, and the bulkheads. An expression for the energy of the external forces is also written. To these expressions, the principle of minimum potential with respect to the amplitudes of the mode shape is applied yielding two simultaneous equations in the mode shape amplitudes. Orthogonality is applied to uncouple these equations. From the resulting equations the critical stress was obtained. The critical stress reduces to the well known value in the case of a thin shell without stiffeners. For each problem it is necessary to minimize the expression for the critical stress with respect to the wave length parameters.

General instability and individual problems arising when restrictions were placed on the deformations of the longitudinal and circular stiffeners (9 cases including general instability) are considered for a specific cylinder with a radius of 100 centimeters and a thickness of 0.4 cm. Data is reduced and presented in tabular form. Calculations are made for various numbers of half wave lengths ranging from 4 to 31 in the tangential direction and from 1 to 63 in the longitudinal direction for the general instability case and two of the restrictive cases. It was generally found that an increase in the length yields a decrease in the critical stress.

Another specific example uses a cylinder with a length of 628 cm., a radius of 100 cm., and a thickness of 0.4 cm. reinforced with 24 longitudinal ribs and from 1 to 3 circular ribs with uniform spacing. General instability and 5 other restrictive cases were considered. Data for the lowest value of critical stress obtained by varying the half wave parameters is presented in tabular form along with the number of half waves in the longitudinal and tangential directions corresponding to the lowest critical stress. Orthotropic results are also compared in tabular form. The results are similar for general deformation (orthotropic theory predicts slightly lower critical stresses) but for the restrictive cases the critical stresses predicted by orthotropic theory are shown to be 40 per cent lower in some cases.

The author is at Academy of Sciences, Ukraine SSR, Institute of Mechanics. 3 References.

2. Amiro, I. Y.: Studying Maximum Load for Ribbed Cylindrical Shells Subjected to Simultaneous Effect of Axial Forces and Internal Pressure. Foreign Technology Division, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio, FTD-TT-62-1625/1 2, January 1963.

A method is presented for determining the critical buckling load for ribbed cylindrical shells subjected simultaneously to axial load and internal pressure. The critical axial load was determined by the energy method. Possible buckling modes are discussed for discrete ribs. An idealized cylinder without ribs is used to describe the deformation shape of the skin.

The elastic stability is first considered by substitution of an assumed buckled mode shape into the compatibility and equilibrium equations to obtain the stress function. An expression for the critical longitudinal stress is derived from the principle of minimum potential. Buckling mode shapes are discussed for the general case and for the cases where the longitudinal or circular ribs, or both, are restricted to bending or twisting. Eight possibilities other than the general case are considered. A table is presented that relates the number of half waves to the cylinder parameters for the general and the eight restricted cases. Extension is made into the plastic region by introducing as the stress the yield stress.

A chart is presented for the critical load as a function of pressure for 27 half waves in the tangential direction and for 50,100 and 150 half waves in the longitudinal direction for a shell with a radius of 4.5 meters, a length of 26 meters, a thickness of 0.5 cm., and 54 ribs. Calculations indicate that the critical axial load is similar to that predicted by orthotropic shell theory. A table is presented that compares the results of this analysis with orthotropic shell results.

The author is at Academy of Sciences, Ukraine SSR, Institute of Mechanics. 5 References.

3. Anderson, M. S.: Combinations of Temperature and Axial Compression Required for Buckling of a Ring-Stiffened Cylinder. NASA TN D-1224, April 1962.

A theoretical analysis is presented for the buckling of cylinders subjected to both axial compressive and thermal stresses for simply supported and clamped end conditions. The basic analysis considers the cylinder walls at a uniform temperature and rings at some lower temperature. Axial compression and temperature combinations necessary to cause buckling are determined by expressing both the variation of circumferential stresses produced by a uniform axial stress and circumferential stresses resulting from the temperature distribution in a Fourier series expansion. Batdorf's modified equilibrium equation is used in the circumferential stress analysis for the case of uniform axial compression. The series expansion for deflection is substituted into the equilibrium expression yielding an infinite stability determinant that is convergent for both clamped and simply supported edge conditions--unlike Donnell's eighth order equation which is possibly divergent in the case of clamped cylinders. In addition to usual small deflection theory assumptions, rings are assumed to be rigid against radial loads but are allowed to expand due to temperature gradients.

Theoretical results are presented in tabular and graphical form and compare favorably with empirical results of another investigation. Results are presented as a buckling temperature coefficient plotted against the cylinder curvature parameter for various values of the ratio of applied axial stress to the classical buckling stress. These interaction charts cover a wide range of cylinder proportions and curvature parameters. Curvature parameters from one to 1,000 are considered and temperature variations from 0°F to 600°F are considered for aluminum 2024 T-3 alloy cylinders.

It is concluded that, for moderate to large values of curvature parameters, the buckling temperature of a cylinder is essentially independent of length and that the buckling temperature in the case where only thermal stresses are considered is beyond the range of use of most materials. It is also determined that a cylinder can endure large changes in axial load without a significant reduction in buckling temperature.

The author is at Langley Research Center.
6 References.

4. Anderson, M. S.: Buckling of Ring-Stiffened Cylinders Under a Pure Bending Moment and a Nonuniform Temperature Distribution. NASA TN D-1513, November 1962.

An experimental investigation is undertaken to determine the effect of axial and circumferential thermal stresses and load-induced stresses on the buckling of cylinders. Experimental results are compared with theoretical results obtained from an analysis presented in the appendix and a former analysis (NASA TN D-1224) for determination of load-induced stresses of buckling.

The analysis presented in the appendix is an extension of the method presented in NASA TN D-1224 by M. S. Anderson. The significant difference is that axial variations in the temperature distribution are accounted for by assuming a constant temperature in each of several bays. Bay length is not necessarily the length between circumferential ring stiffeners but is dependent on the desired accuracy in representation of the axial temperature distribution.

Thirteen cylinders with type 301 stainless steel walls and type 304 stainless steel spun Z-section rings were tested in pure bending. All cylinders were 19 inches in diameter and 45 3/4 inches long with a nominal wall thickness of 0.030 inches and a resultant value of radius to thickness ratio of approximately 300. Rings were 1¼ in Z-sections with 3/4 inch flanges. Two ring spacings were used with spacing to radius ratios of 1 and of ½. Cylinders were fabricated of sheet material with 3 longitudinal spot welded splices. Specimen details are presented in drawings.

Loading was accomplished by mounting the specimen to a heavy backstop and applying a pure bending moment by means of a pin-connected loading frame. Precautions were taken to eleminate shearing forces on specimens. Two cylinders with a ring spacing to radius ratio of ½ and one cylinder with a ring spacing to radius ratio of 1 were tested in pure bending at room temperature to obtain data for comparison. The significant difference between the tests at room temperature and the tests with heating is that in the case of the room temperature tests, the load was applied by means of hydraulic jacks rather than by a pin-connected loading frame. Tests with heating were accomplished by loading the specimen in bending to some load less than the classical buckling load and heating the structure at a rate of 25° F/second until buckling occurred. The vertical deflection at the end of the cylinder was recorded continuously to determine the instant of buckling. End moments were varied and a buckling interaction curve for load-induced stress and temperature was obtained.

Results of the investigation are given in tabular and graphical form. Maximum load-induced stress is plotted against the temperature of the extreme compressive fiber between stiffener rings for both uniform and nonuniform heating. Temperature distributions are also plotted.

Results indicate that elementary thermal stress theory is inadequate for prediction of buckling loads of cylinders under nonuniform heating. An analysis is presented that shows reasonable correlation with experiment.

The author is at Langley Research Center.
5 References.

5. Ando, N.: On the Strength of the Orthogonally Stiffened Plate--1st Report--Theoretical Solution of Orthotropic Plate Subjected to Bending, Ministry of Transportation, Tokyo, Japan, March 1962.

Orthotropic plate theory is applied to the analysis of orthogonally stiffened rectangular plates. Three approximate methods are considered --the grid structure method, the energy method, and orthotropic plate theory. The basic differential equation is presented and the form of solution is discussed with particular attention paid to boundary conditions.

A theoretical solution to the fundamental differential equation is obtained as the sum of the general and particular solutions. The form of the particular solution is discussed for various types of loading--uniformly distributed load, hydrostatic pressure, concentrated line load, and symmetrically positioned concentrated loads. For simply supported edge conditions, solutions are derived for a distributed load in a rectangular domain, a line load, a concentrated load, a moment distributed along an axis of the plate, and a distributed loading over a general area on the plate. For the case of two edges simply supported and two built in, solutions are obtained for a distributed load in a rectangular domain, a line load, and a concentrated load. For the case of two edges simply supported and a moment distributed along the other, both symmetric and antisymmetric moment distribution solutions are obtained. A solution is also obtained for the case of three edges simply supported and the other free.

A solution is obtained for the case of two edges fixed and a moment distributed along the other edges. Exact and approximate solutions obtained for the case where all edges are fixed. Deflections at the center of the plate are calculated by the exact and approximate methods and are presented in tabular form for plates of various aspect ratios.

A solution is presented for an orthotropic plate partly stiffened by a large stiffener. Further extension of theory is presented for the case of continuous plates. Deflection distribution along the longitudinal of a continuous plate is shown graphically for various load ratios. Theory is also extended to a rectangular box composed of orthotropic plates.

The author is at Ministry of Transporation, Tokyo, Japan.
54 References.

6. Armenakas, A. E., and Herrmann, G.: On the Buckling of Circular Cylindrical Shells Under External Pressure. Columbia University, Project 7?7, Contract AF 49 (638)-430, Technical Note 7, August 1962.

A bending theory previously presented by the authors is employed to determine the values of hydrostatic and constant-directional pressure to cause buckling of circular cylindrical shells. Inadequacy of equations previously developed by other authors is primarily attributed to the negation of the change in direction and magnitude of applied pressures resulting from deformations.

Changes in pressure, forces, and moments are expressed in terms of a non-orthogonal set of unit vectors and unit elongations. The non-orthogonal set of unit vectors arise from the orthogonal set of unit vectors describing a surface element undergoing deformations. Solutions of deformations are then assumed and introduced into the equilibrium equation resulting in three homogeneous algebraic equations whose conditional solution yields the buckling pressure. Underlying boundary conditions for the assumed deformation solutions are discussed. Equations for the buckling coefficient are then presented for both the hydrostatic and constant-directional pressure cases. A discussion follows relating to the terms in the buckling coefficient equations. It is shown that a shell cannot buckle under external constant-directional pressure but may buckle under external hydrostatic pressure.

Approximate formulas are presented resulting from consideration of the number of axial half waves. It is shown that unless the shell dimensions are conducieve to buckling, the effect of the nature of the applied pressure is negligible. Charts are presented that allow the determination of the mode for the cylinder parameters: length to radius ratio and thickness to radius ratio. The charts are needed for use in the equations for the buckling coefficient.

Theoretical results of other investigators are compared graphically and it is shown that the equations of von Mises, Loo, and Batdorf are applicable for various ranges of shell parameters.

The authors are at Columbia University.
14 References.

22. Becker, H.: Handbook of Structural Stability, Part VI-Strength of Stiffened Curved Plates and Shells. NACA TN 3786, July 1958.

A comprehensive review of the theories of instability failures of plates and shells is presented. General instability of circular cylinders is discussed where loadings considered are bending, external pressure, torsion, transverse shear, and combinations of these loading conditions. The primary objective of the report is the examination of the methods of predicting bending and torsional general instability failure in stiffened circular cylinders.

The theory of Taylor is used as the basis for analysis. Lack of agreement of previous papers on general instability is attributed to the evaluation of pertinent section properties and rigidities. The approach of Hoff in which the energy increment is minimized is also discussed. Experimental results from tests conducted at Guggenheim Aeronautical Laboratory, California Institute of Technology (GALCIT) and the Polytechnic Institute of Brooklyn, Aeronautical Laboratory (PIBAL) form the basis of an empirical approach presented. Test data from GALCIT and PIBAL are presented in graphical form. Effects of plasticity on failure stress are considered and a reduced modulus (secant modulus) is proposed.

Minimum weight design is discussed using the mathematical formulation of Gerard. Fundamental concepts of minimum weight analysis in general are reviewed and the application to stiffened cylinders in bending is presented where the minimum weight conditions of a stiffened cylinder in bending are stated.

Theoretical and analytical results of PIBAL for stiffened circular cylinders with cutouts is presented. Theoretical analysis follows the energy approach where the cutout is included in the analysis by representing the buckled form as a sine wave extending the length of the cutout and by a Fourier series (7 terms) extending in the circumferential direction. PIBAL theory predicts the critical moment accurately for symmetric cutouts but the instability stresses predicted range to 35 per cent greater than tests indicate. An attempt to predict instability in the case of side-cutouts failed.

A section of this report is devoted to pressure instability. Available theoretical data are collected since no tests are currently available. Moderate-length and long circular cylinder theoretical developments are presented.

Problems of effective widths and appropriate section parameters to be used in the equations are discussed. Data for all cases are presented graphically.

Hayashi's method for torsional instability covering the entire length range is discussed. Hayachi used an implicit form of instability and in this report explicit data are presented in tabular form. A discussion of pertinent section properties along with effective widths for torsion is presented. GALCIT test data, applicable to moderate length ranges, are used as a basis for comparison.

No formal analysis is presented for the case of transverse shear instability due to the lack of a published theory. GALCIT data for cantilevered torsionally loaded cylinders are applied to obtain a conservative (15 per cent) empirical expression.

Combined bending and torsion are treated by interaction equations. Interaction curves that are parabolic and circular are found to include the test data and serve as upper and lower bounds. Test data and analytical data are presented graphically.

The author is at New York University.
56 References.

8. Becker, H. and Gerard, G.: Elastic Stability of Orthotropic Shells. Journal of Aerospace Sciences, Vol. 29, No. 5, May 1962, pp. 505-513, 520.

Objectives are twofold: (1) to correlate available experimental data on elastic stability of orthotropic shells that fail due to external pressure, torsion, or axially compressive loading; and, (2) to use the Batdorf modification of the Donnell eight order equilibrium equation to obtain solutions for zero length to long range orthotropic cylinders covering a wide range of curvature parameters.

Possible buckling modes, characteristics of attachment of the cylinder-stiffener combination, and the theoretical treatment of the skin stiffening system are discussed. Taylor's governing differential equation is presented based on elastic behavior, linear small-deflection theory, and a Poisson's ratio of zero. The underlying assumptions of the Taylor differential equation are discussed. Also included in the preliminary discussion is the effect of pressurization on the strength of cylinders with initial imperfections. Distinction is made between the actual and the geometric stiffnesses of cylinders. In the case of axially compressive loading, this distinction becomes of vital importance.

The approach used assumes a reasonable solution to the Donnell equation (primarily dictated by boundary conditions) and obtains the buckling coefficients as a function of buckle mode parameters and curvature parameters for each loading case by differentiation as dictated by the Donnell equation. The expression is then minimized for the case of a flat plate (curvature parameter of zero) for appropriate length ranges. Due to mathematical complexities the buckling coefficient is necessarily minimized numerically in the short cylinder range. The results are plotted against a unified set of non-dimensional isotropic shell parameters for cylinders of varying length.

Results of the theoretical investigation are compared with experimental findings from other investigations. Both orthotropic and isotropic cylinder data compare favorably with theoretical data for pressure instability. Torsional instability results are compared with isotropic cylinder theoretical results by Batdorf. For the axially compressive case, the experimental data are severly limited and only one point is available for comparison. This information is obtained from an experimental investigation using a 2014-T6 aluminum alloy cylinder 7 feet in diameter, 4 feet in length, and with a 0.055 inch skin thickness. The cylinder was integrally stiffened with rings that have a depth of 0.125 inches and a thickness of 0.195 inches. The rings were spaced at 1.75 inch intervals. Agreement, in this case, is excellent but further experimental investigation is deemed necessary.

The authors are at New York University.
18 References.

9. Becker, H., Gerard, G. and Winter, R.: Experiments on Axial Compressive General Instability of Monolithic Circumferentially Stiffened Cylindrical Shells. New York University, Research Division, Technical Report No. SM 62-5, May 1962.

An experimental program is conducted on machined orthotropic aluminum alloy cylinders with ring stiffeners under axially compressive loading to investigate the general instability characteristics of stiffened cylinders. Linear orthotropic stability theory is used to predict the buckling load and mode shape for moderate length cylinders.

Care is taken to assure that no mode other than the general instability mode occurs during testing. Tests were conducted on two isotropic and twelve ring stiffened circular cylinders loaded in axial compression. The cylinders were nominally 8 inches and 24 inches in diameter. An analysis performed on a 7 foot diameter cylinder is also included. Variations in ring spacing, ring details, and shell thickness were incorporated in the specimens. All cylinders were made of 2024-T3 aluminum alloy except two specimens that were made of 2014-T6 aluminum alloy. Geometric, structural, and theoretical and experimental buckling stress data are presented in tabular form for all cylinders. Six of the cylinders were formed by threading rings on a lathe to obtain a more uniform cylinder wall and rib geometry. Ends of the cylinders were faced to a maximum variation of 0.0005 inches throughout the end planes. Ends of one model were encased in transparent epoxy rings for photoelastic analysis. No significant variations in fringe patterns are observed in the test. Details of the three end conditions used are presented.

Data are presented in terms of both the average measured thickness and the minimum measured thickness. Due to uncertainty as to which value should be used in theoretical analysis, structural parameters (dimensionless length and buckling coefficient) are calculated for both average and minimum thicknesses. Structural mode parameters are calculated for only the average thickness. In addition to the general instability load, data was obtained for buckle geometry and post-buckling behavior of each specimen. Photographs of the post-buckled state of the cylinders are presented.

The cylinders investigated failed in general instability with no other mode present. Data indicate that linear theory is adequate for prediction of the load carrying capacity of monolithic ring stiffened circular cylinders.

A lower limit of application of linear orthotropic theory is tentatively set in terms of the two structural parameters obtained from asymmetric theory. The transition from isotropic to orthotropic behavior in the region not considered in this program still requires investigation. Excellent agreement with theory is obtained from both 24 inch and 8 inch diameter cylinders indicating that no significant size effect is associated with the circumferentially stiffened cylinders tested.

The authors are at New York University.
7 References.

10. Berkowitz, H. M.: Elastic Deformations of Conical Shells, Equation of Equilibrium for Large Elastic Deformations of Relatively Thin Turncated Conical Shells. Fairchild Stratos, Aircraft Missles Division, FS-AMD, R 62-1, September 1962.

Determination of approximate equations governing the large-deformation behavior of relatively thin, truncated, elastic conical shells subjected to arbitrary loading on all surfaces is considered.

Discussion is restricted to elastic materials whose stress is derivable from an elastic potential. The form of the elastic potential used is the same as that of linear small deflection theory. Using the elastic potential, stress is expressed in terms of strain. The relation is shown to reduce to the isotropic stress relation for an elastic medium. For the purpose of illustration only isotropic materials are considered further.

The author is at Fairchild-Stratos.
12 References.

11. Card, M. F. and Peterson, J. P.: On the Instability of Orthotropic Cylinders. NASA TN D-1510, December 1962, pp. 297-308.

Preliminary results of an experimental investigation of the buckling strength of laterally and longitudinally stiffened cylinders and filament-wound glass-epoxy cylinders are compared with instability calculations based on small deflection orthotropic cylinder theory.

Calculations are performed to study the effect of certain stiffness parameters on calculated buckling loads for the laterally and longitudinally stiffened cylinders. From these calculations it is determined that, throughout the range of uncertainty of the stiffness parameters, the extensional modulus in the circumferential direction and the twisting stiffness are not the cause of discrepancy in calculated buckling strengths. It is thus determined, by elimination, that the source of the discrepancy is the estimated values of wall stiffness in bending and in wall shearing stiffness. Another calculation is performed to determine the likelihood of panel buckling as a source of the discrepancy. It is found that the cylinder with the largest lateral stiffener spacing may have failed in panel buckling but that this was unlikely in the case of the other cylinders.

Laterally and longitudinally stiffened cylinders were tested in bending. Cylinders were 77 inches in diameter and were stiffened with Z-section stringers and hat-section rings. Glass epoxy cylinders were 15 inches in diameter and tested in axial compression. Epoxy cylinders were constructed with both circumferential and helical windings to achieve orthotropic properties.

Stiffnesses of the filament-wound glass-epoxy cylinders are determined from the equations of elasticity for orthotropic materials. The epoxy wall experienced plastic deformation that subsequently lowered the buckling load for helical winds of 45° to $67\frac{1}{2}^{\circ}$ but for a helical wind of 25° the cylinders appeared undamaged by plastic deformation. The 25° helically wound cylinders did experience lower buckling loads on reloading. Excellent agreement between theory and experiment is attributed to the low radius to thickness ratio (in the order of 125) of the 25° helically wound cylinders. Previous studies indicate buckling loads of cylinders with low radius to thickness ratios deviate little from theory.

It is concluded that, in the case of laterally and longitudinally stiffened cylinders, wall stiffness in shear and the circumferential bending stiffness need better definition. With respect to the glass epoxy cylinders, the mode of failure needs better definition to determine significant stiffness and plastic reduction parameters. Furthermore, even small rings affect the instability mode and further testing is necessary to establish a quantitative effect on panel instability.

The authors are at Langley Research Center.
9 References.

12. Connor, J. J., Jr.: Elastic Buckling of Axially Compressed, Thin, Unreinforced Cylindrical Shells. Watertown Arsenal Laboratories, WAL TR 836.3212, January 1960.

A survey of literature pertaining to the behavior of axially compressed, thin, unreinforced cylindrical shells is presented.

For unpressurized shells, small deflection or classical theories by Donnell whose work is extended to edge warpage in the axial direction and Batdorf whose work using the Galenkin approach to Donnell's modified equation includes clamped and edges free to warp in the circumferential direction are presented and discussed with emphasis on the validity and applicability of underlying assumptions.

Also, for unpressurized shells, semi-empirical approaches are discussed. The work of Batdorf is briefly discussed. A semi-empirical method presented by Harris et. al. similar to the work of Batdorf is discussed at length. All length ranges including short, intermediate, and long are discussed for cylinders with clamped edges. Cylinders with simply supported edges are also included in the discussion. Design charts are presented in which semi-empirical and small deflection theories are compared.

Post buckling behavior of unpressurized shells is also discussed. Original work by Donnell is discussed. An extensive discussion of the work by von Kármán and Tsien is presented and the derivation of equilibrium and compatability equations is presented in the appendix. Effects of imperfections are discussed along with a brief discussion of classical finite deflection stability criterion.

For pressurized shells, a finite deflection stability criterion formulated by Lo, Crate, and Schwartz is discussed along with their experimental observations. A discussion of the effect of internal pressure on the buckle mode is included along with a brief discussion of plastic buckling of shells. Results of experimental investigations by Harris et. al. and Fung and Sechler are presented graphically. Semi-empirical analyses are discussed and theoretical and recommended design data are presented graphically.

The author is at Watertown Arsenal.
22 References.

13. Findley, W. N.: Theories Relating to Fatigue of Materials Under Combinations of Stress. Engineering Materials Research Laboratory, Division of Engineering, Brown University, U. S. Army Ordinance Corps, Contract DA-19-020-ORD-3520, Project [illegible] (1348), Technical Report No. 2, June 1956.

The mechanism of fatigue of materials is discussed. Some of the primary factors that induce resistance to fatigue fracture are discussed in elastic and plastic loading regions. The origination of fatigue cracks is explained for every combination of stresses by a single fatigue mechanism theory using as the basic hypothesis the transition from shearing slip to tensile separation including the effects of anisotropy of the material.

Effects of static stresses are discussed extensively considering the effects of tension and torsion. A series of three tests on 75-S-T aluminum is mentioned consisting of a test at zero mean stress, a test that initially induces yielding and a load reversal to the minimum cyclic stress, and a test at a high mean stress using the same approach as is used in the second test.

Six of eight theories for combined bending and torsion are reduced to a single expression which is equivalent to the Gough empirical ellipse quadrant. The principle stress and strain theories are reduced to parabolic and elliptic areas respectively.

Fatigue tests are reported that explored the influence of anisotropy. Tests were made on three alloys (76S-T61, 25S-T6, and SAE 4340 steel). The tests support the conclusions of the effect of anisotropy. Variations in the bending to torsion stress ratio are explained with the use of anisotropy, complementary normal stress, and the orientation of the shearing stress with respect to the material texture.

The author is at Brown University.
8 References.

14. Fitzgibbon, D. P.: Experimental Method for Testing Materials in Biaxial Stress Fields. Space Technology Laboratories, Los Angeles, STL / TR-60-0000-09028, February 1960.

An experimental procedure for investigating the biaxial stress states that arise in pressurized vessels is presented. The main objective of the report is to set down an experimental procedure with sufficient flexibility to be extended to investigations of any tension-tension biaxial stress field. This report is a preliminary investigation and contains only a discussion of the experimental procedure with no results presented.

A limited description of biaxial stress fields is presented. The experimental procedure is basically to maintain a constant ratio of longtudinal to circumferential stress while increasing these stresses to failure. This is to be carried out for several stress ratios until sufficient data is obtained. Direct axial loading is accomplished with a testing machine while pressure loads are applied with the use of a hydraulic jack. X - Y plotters are used to record longitudinal and circumferential strains as a function of pressure. A detailed list of equipment is provided in the appendices along with a block diagram of the instrumentation.

Data are electronically reduced and corrections are discussed. It is shown that although the data obtained must be reduced, initial test records serve as useful comparative results since data corrections are usually comparatively small.

The author is at Space Technology Laboratories
5 References.

15. Compressive Stability of Orthotropic Cylinders. New York University, College of Engineering, Research Division, Technical Report SM 62-4, May 1962.

Elastic and plastic buckling of short isotropic and orthotropic cylinders loaded in axial compression is considered. The governing differential equation is obtained from equilibrium and strain-displacement conditions. Solutions for the buckling coefficients are obtained for both axisymmetric and asymmetric modes. The buckling coefficients are minimized for the moderate length ranges.

The behavior of the solutions over the length ranges corresponding to flat plates, short cylinders, and moderate length cylinders is examined for the elastic case. In the moderate length range, solutions are presented graphically in parametric form. Additional charts of buckling coefficient ratio vs. wave length parameter are presented. Upper limits for the short cylinder region are determined for both the axisymmetric and asymmetric modes in terms of a curvature parameter and the buckling coefficients. The results for the short cylinder region are presented in graphical form.

For the plastic case, solutions are obtained for the axisymmetric and asymmetric modes. Solutions for wave length parameters and buckling coefficients are presented graphically and are compared with available isotropic test data for moderate length aluminum alloy cylinders (3003-O, 2017-T4, and 7075-T6) to establish limits on the ranges of the explicit solutions presented. A plasticity reduction factor is introduced and plotted against curvature. For orthotropic plastic cylinders, buckling coefficient vs. curvature charts are presented. Effects of plasticity on moderate length orthotropic cylinders are demonstrated. A quantitative illustration is used to calculate the buckling coefficient vs. curvature charts.

Post-buckling behavior of orthotropic cylinders is qualitatively discussed. A schematic representation of the behavior of moderate length cylinders is presented graphically and the special case of the isotropic cylinder is discussed for both elastic and plastic cylinders. The plastic analysis is based, in parts, on the analysis of Lee. Effects of initial imperfections are qualitatively discussed.

The author is at New York University.
7 References.

16. Gerard, G.: Elastic and Plastic Stability of Orthotropic Cylinders. NASA TN D-1510, December 1962, pp. 277-295.

Elastic and plastic buckling behavior is predicted for orthotropic cylinders of moderate and short length by linearized stability theory. Plastic buckling behavior is investigated for isotropic cylinders in the same length ranges. All of the theoretical results are presented graphically and correlated with experimental data. Comments are generally confined to cylinders loaded in axial compression although mention is made of the general agreement in buckling behavior of other loading mechanisms with linearized theory.

In the case of moderate length orthotropic cylinders, results indicate that the failure mode is influenced by the buckling coefficient ratio and the wave length parameter. For buckling coefficient ratios greater than one, results indicate that failure is governed by the axisymmetric mode. For buckling coefficient ratios less than one, or for imaginary wave length parameters, the asymmetric mode is found to govern. For short cylinders the buckling mode is determined by the curvature parameter. Axisymmetric modes are found to govern only for curvature parameters less than 1.4.

Axisymmetric modes of failure are found to govern for moderate length isotropic cylinders in the plastic region. Plasticity reduction factor equations are presented for axisymmetric and asymmetric modes for short isotropic cylinders in the plastic region. The axisymmetric solution for isotropic cylinders is shown to be applicable to orthotropic cylinders in the plastic region. For the asymmetric modes, it is found that the proper orthotropic plasticity coefficient must be used.

Correlation with available experimental data is made for orthotropic elastic cylinders. The author concludes that orthotropic construction is more reliable than isotropic since prediction of buckling loads is more accurate. For failures of plastic cylinders in the axisymmetric mode, correlation is excellent. However, asymmetric failures exhibited discrepancies with theory.

Post buckling behavior is briefly discussed along with the effect of imperfections in the cylinder. Discussion reveals that a decided effect on the buckling characteristics in elastic cylinders but a negligible effect for plastic cylinders.

Author concludes that structural reliability of orthotropic laterally stiffened cylinders exceeds that of longitudinally stiffened isotropic cylinders.

The experimental program was conducted with only laterally stiffened orthotropic 2014-T6 aluminum alloy cylinders. Results from another NASA sponsored investigation are used for 7075-T6 and 3003-0 aluminum alloys (NASA Research Grant NSG-17-59 with New York University).

The author is at Allied Research Associates.
9 References.

17. Gibson, J. E.: Computer Analysis of Thin Shells. Symposium on the Use of Computers in Civil Engineering. Laboratorio Nacional de Engenharia Civil, Paper No. 24, October 1962.

Programs are described for determining moment and stress resultants for general multi-shell structures without edge beams that are capable of solving shells with up to thirty bays.

Two basic-procedures are followed. The first method is derived from equilibrium and compatability relations where the governing equation is presented along with the general and particular solutions involving eight arbitrary constants to be evaluated from the boundary conditions. The second method is derived from displacement considerations. Three partial differential equations arise and an assumed solution yields a characteristic equation where the vanishing determinant of the coefficients furnishes the needed information for calculation of stress and moment resultants.

Boundary conditions are found to require that the stress and moment resultants must vanish at the free edge; that the displacement be continuous at shell intersections; and that the moment and stress resultants be equal at shell intersections.

Example applications to asymmetrical shells are cited. The first pertains to a shell with a total arc angle of 80°, a radius of 30 feet, a length of 120 feet, a thickness of 0.25 feet, and a total loading angle of 20°. Data is presented in tabular form. The second pertains to wind loading on a double cylindrical shell of the same specifications as the first problem with a wind loading of 15 psf. Results are presented in graphical form. The third problem pertains to the wind loading of a flat roof in which small arc angles (2°) are used to approximate a flat surface. The radius used is 200 feet in this case. The angle of inclination of the shells is 30°. The results of the analysis of a shell of double curvature are presented in tabular form. The shell has the same overall geometry as that of problems one and two.

The author is at University of Manchester.
6 References.

18. Hodge, P. G., Jr.: Plastic Analysis of Circular Conical Shells. Department of Mechanics, Illinois Institute of Technology, DOMIIT Report No. 1-8, August 1959.

Basic equations necessary to predict the collapse load of a right conical shell when the load is applied over a finite area at the vertex are reviewed. Shell dimensions and stress resultant and displacement sign conventions are shown. Equations of equilibrium along a generator and moment equilibrium equations are written in dimensionless form. Generalized strain rates and generalized stresses are then written in terms of displacements. Boundary conditions are stated and external and internal energy dissipation rates are presented. The fact that the external and internal energy dissipation rates must be equal is stated to provide an upper bound on the load.

When the load is applied over a finite area, two problems are considered--one involving direct stress and the other involving bending moments. A solution is presented for the direct stress. A moment distribution is also proposed but found not to be statically admissable. A new hypothesis is formed and another moment distribution results that is statically admissable. Applying the plastic flow law to the assumed stress profile and using the boundary conditions, a kinematically admissable velocity field is derived. Internal and external energy rates are then equated to yield the collapse load. The equation presented for the collapse load is shown to provide upper and lower bounds for the collapse load for all shells.

For the case of a concentrated load the static problem is approached directly by letting the area of load application approach zero in the previously derived equations. The velocity field is approached indirectly since if the area is reduced to zero an inadmissible velocity field is created. The equations are shown to apply to shells with other edge supports.

A more direct approach to the problem of a general rotationally symmetric shell is presented and is illustrated for the case of a shell cap. Solutions of shells of medium flatness are presented in the appendix.

The author is at Illinois Institute of Technology
6 References.

19. Hodge, P. and Panarelli, J.: Plastic Analysis of Cylindrical Shells Under Pressure, Axial Load and Torque. Illinois Institute of Technology, Department of Mechanics, DOMIIT Report No. 1-17, April 1962.

An explicit parametric form of the yield curve is obtained for thin shells that yield according to the von Mises yeild criterion. For arbitrary shells, the yield condition is expressed in terms of six generalized stresses. In the case of the cylindrical shell, the variables are reduced to four. Generalized stresses are further reduced to two by restricting the loading mechanism to the case of a cylindrical shell with axial and twisting loads applied only at the ends.

The case of the cylindrical shell loaded only at the ends is treated with the use of dimensionless stress parameters. The resulting yield curve is shown to be bounded from the interior by its inscribed circle and from the exterior by a circle of 5 per cent greater radius than the inscribed circle. It is concluded that a solution according to a circle provides close bounds on the interaction curve. Simplified solutions for singular portions of the yield surface are discussed briefly. Discussion is limited since only the bounds of solution are of primary concern. A load parameter is introduced that differs from the original stress parameters in that it is related to the yield stress of the material in simple tension rather than the stress to cause yield in the structure. Upper and lower bounds in terms of a load parameter are furnished to describe the yield condition.

Examples are cited for the special case of an applied torque and pressure with no end load and an ordinary pressure vessel where the pressure acting on the ends of the vessel causes an axial load. Interaction curves are plotted for these special cases. The dimensionless parameters are again related to the yield stress in simple tension of the material and not to the load that causes yield in the structure.

The Appendix provides an approximate solution for the yield curve.

The authors are at Illinois Institute of Technology.
3 References.

20. Hom, K.: Elastic Stresses in Ring Frames of Imperfectly Circular Cylindrical Shells Under External Pressure Loading. Structural Mechanics Laboratory, Research and Development Report No. 1505, May 1962.

A solution for the elastic stresses in the ring frames of imperfectly circular cylindrical shell of finite length subjected to external pressure presented by Kendrick is reviewed. The Kendrick analysis is developed with the principle of minimum potential energy (Rayleigh-Ritz method) assuming an initial imperfection of the same form as buckling displacements. The assumption of the initial imperfection parameter presented by Kendrick is shown to be improbable and a more realistic form is presented.

Kendrick's general approach is closely followed with modification only in the imperfection parameter used. The potential energy is written as a function of the energy stored in the frame and shells and the total work done by the external load. Strain relationships are introduced (derived in the Appendix) as a function of the buckling and axisymmetric load displacements. The principle of minimum potential energy is applied to obtain a set of linear non-homogeneous equations in mode shape parameters as a function of the initial imperfection amplitude. Mode shape parameters introduced are functions of buckling displacements. Finally, the elastic strain equation as a function of the buckling displacements is introduced (derived in Appendix). An approximate solution is obtained by using the relations from the complete solution for the special case of an infinite cylinder subjected to uniform external pressure neglecting small-order terms. The approximate and complete solution compare favorably.

Tests are currently being conducted to confirm the theory presented. Preliminary results are obtained using cylinders constructed of high-strength steel with a yield strength in the range of 45,000 to 50,000 psi. Stiffeners were made of T-frames with an overall length about 1.6 times the cylinder diameter between rigid ends. One small cylinder was constructed as a comparison cylinder with no initial imperfection. Two other specimens were of welded frame construction with an initial imperfection. The two test structures had different magnitudes of imperfection.

Results thus far gathered experimentally are compared with analytical findings. It is found that the approximate and complete solutions presented compared more favorably with experimental results than the Kendrick analysis. All results are presented graphically.

The author is at David Taylor Model Basin.
16 References.

21. Keller, H. B. and Reiss, E. L.: Some Recent Results on the Buckling Mechanism of Spherical Caps. NASA TN D-1510, December 1962, pp. 503-514.

The buckling mechanism of clamped spherical caps is qualitatively discussed. Initial load parameter, versus maximum deformation curves, first proposed by von Kármán and Tsien, are discussed. The curve von Kármán and Tsien presented indicates that the spherical cap can exist in a maximum of three equilibrium states. The possibility of more than three equilibrium states existing simultaneously is discussed. The buckling mechanism is described by ordering the equilibrium states existing simultaneously in terms of the potential energy in each state. The triggering mechanism that causes the dynamic transition from one equilibrium state to another is discussed. Only axisymmetric deformation states are considered but the transition problem is not limited to the axisymmetric case.

Bifurcation buckling theory is used to bracket and yield close bounds for the intermediate buckled state where the energy in this state is equal to the energy in the lowest buckled state (lowest load parameter). A qualitative load-deformation curve for bifurcation buckling is presented.

For relaxation buckling the method of Murray and Wright is used. Only preliminary results are summarized relating to the number of equilibrium states and the energy loads. Equilibrium states are discussed for a large range of geometric parameters. Unbuckled, buckled, and "triggering" states are discussed in the geometric parameter ranges considered. The largest number of multiple solution thus far determined is nine.

Dependency of the problem on boundary conditions is discussed. Edges are considered clamped in this report. A procedure is proposed to eliminate the effects of initial imperfections in experimental tests.

The authors are at New York University.
10 References.

22. Kuenzi, E. W.: Buckling of Layered Orthotropic and Sandwich Cylindrical Shells in Axial Compression. NASA TN D-1510, December 1962, pp. 323-330.

Results of the modification of the von Kármán and Tsien large deflection theory for the calculation of critical loads as applied to orthotropic cylinders loaded in axial compression are presented. Results are also presented for the modification of a theory for determing the critical loads of sandwich cylindrical shells loaded in axial compression which was previously presented by the author.

Extensive experimental data previously determined for plywood cylinders constructed of yellow birch and yellow-poplar veneers are compared graphically with analytical predictions of critical loads. Only limited comparisons of sandwich core data are presented. The sandwich sheets were made with aluminum facings and cores of end-grain balsa wood and cores of soft cork board. Precautions were taken to assure that the critical load was below the compressive proportional limit stress.

Correlation between analytical and experimental data is only fair due to the large amount of scatter in the experimental data.

The author is at United States Forest Products Laboratory.
8 References.

23. Lu, S. Y. and Nash, W. A.: Elastic Instability of Pressurized Cylindrical Shells Under Compression or Bending. University of Florida, Technical Report No. 1 for Research Grant NSG-16-59, January 1962.

A theoretical analysis of pressurized cylindrical shells in bending or axial compression is presented using nonlinear finite deflection theory. The compatibility and equilibrium equations are expressed in terms of the Airy stress function for membrane stresses. An approximate form of the deflection pattern is assumed and an expression for the Airy stress function is proposed. An approximate solution of the equilibrium and compatibility equations is obtained using Galerkin's method.

Due to the form of the proposed stress function it is possible to obtain solutions for the cases of bending and compression separately using the same approach. Results are presented graphically as critical stress vs. pressure and compared with the results of Lo and Thielemann and the experimental data of Fung and Lofblad in the case of axial compression. For eccentric compression, or pure bending, the results are compared with the experimental results of Suer, Harris, Skene, and Benjamin for axial compression. All results are presented in non-dimensional form.

The effect of initial imperfections is discussed. It is assumed that the ratio of the incremental change in critical stress to the critical stress at zero pressure ratio is practically constant with respect to imperfections. This ratio is plotted against a non-dimensional pressure allowing test data to be obtained at only one pressure and the value of critical stress in the same imperfect shell to be calculated at any other pressure. Test data support the validity of this concept.

The theory presented is in good agreement with test data for a wide range of pressures for both axial compression and pure bending. However, the values of critical stress are conservative for values of dimensionless pressures greater than 0.1.

The authors are at University of Florida.
9 References

24. Morgan, W. C. and Bizon, P.: Experimental Investigation of Stress Distributions Near Abrupt Change in Wall Thickness in Thin Walled Pressurized Cylinders. NASA TN D-1200, June 1962.

An experimental investigation is conducted to determine the validity of previously published procedures in predicting the stress distributions arising from discontinuities of the middle surface of thin-walled pressurized cylinders. A modified analysis as applied to both thin and moderately thick-walled cylinders is presented in the Appendix. The principle of superposition is applied by adding the stresses obtained as a result of surface discontinuity directly to the membrane stress.

Experimental data was obtained for large cylinders with diameter to thickness ratios of 117 and small cylinders with diameters to thickness ratios of 28. Large cylinders were made of a single sheet of 6061-0 aluminum alloy rolled to shape and welded on a single longitudinal seam. Flanges were welded to the structures and the entire structure was heat treated to the T6 condition. Chemical milling was used on the large cylinders to produce thickness changes in the walls. Small cylinders were machined of 2014 - T6 aluminum extruded tubing. Abrupt changes in wall thickness caused by machining offered closer agreement with analytical assumptions. Tests were made at internal pressures of 100, 110, 120, 130, 140, and 150 psi for the large cylinders and 200, 300, 400, 500, 600, and 623 psi for the small cylinders.

All cylinders were constructed with a ratio of wall thickness to change in wall thickness of 0.4. Data obtained from tests are compared with analytical results graphically. The maximum variation between experimental and analytical results occur in large cylinders and is thought to have resulted from the disparity of the actual structure with analytical assumptions. A maximum variation of 10 per cent is found for the large cylinders and 6 per cent for the small cylinders.

It is concluded that stresses arising from the discontinuity of the middle surface of pressurized cylinders have a marked effect on the stress field. This effect is most apparent in the case of a continuous outer surface and discontinuous inner surface.

The authors are at Lewis Research Center.
4 References.

25. Mushtari, K. M. and Schenkov, A. V.: Stability of Cylindrical and Conical Shells of Circular Cross Section, with Simultaneous Action of Axial Compression and External Normal Pressure. Translation of "Ob ustoichivosti tsilindricheskikh i konicheskikh obolochek krugovogo cecheniia pri sovmestnom deistvii osevogo szhatiia vneshnego normalnogo davleniia." Prihladnaia Mathematika i Mekhanika, Vol. 18, No. 6, November-December 1954. NACA Technical Memoranda 1433, April 1958.

A theoretical analysis is given for the determination of the upper limit of the critical load for circular cylindrical and conical shells. Loading consists of the simultaneous action of compression uniformly distributed over a cross section and external normal pressure.

Differential equations of equilibrium are written in terms of a stress function and a normal displacement. Boundary conditions are written for the case of simply supported ends. By means of a change of variable substitution the differential equations and boundary conditions are rewritten in a form more adaptable for solution. A normal displacement function is assumed reducing the equations to a fourth order equation in the stress function which is solved. Boundary conditions are introduced making possible the approximate evaluation of the four arbitrary constants.

The characteristic equation is obtained (by the Bubnov-Galerkin method). The complex characteristic equation is simplified by the assumption of a thin shell. The square of the normal displacement frequency is assumed much greater than unity resulting in an equation for the critical load. The critical compressive load is then written for the case of zero pressure and does not deviate from the exact analysis of Shaterman by more than 5 per cent (conservative) for thin shells.

An expression for the critical isotropic external pressure is obtained. For the special case of the cylindrical shell, the critical pressure is obtained with the shell under the simultaneous action of a specific compressive load. The resulting expression is shown to provide an upper limit to the critical pressure.

Russian Translation.
6 References.

26. Nachbar, W. and Hoff, N. J.: The Buckling of a Free Edge of an Axially Compressed Circular Cylindrical Shell. Quarterly of Applied Mathematics, Vol. XX, No. 3, October 1962, pp. 267-277.

The classical linear equations for axially compressed circular cylinders are solved for free-edge boundary conditions. An explanation of why the loaded circular edges of the shell buckles before the other edges if they are not reinforced is presented.

The classical equations for displacements are used in dimensionless form and are based on the assumptions of Donnell. These equations are presented as functions of the dimensionless stress function and dimensionless displacements. Expressions for the total edge resultant forces acting on the surface cut out by plane normal to the longitudinal axis of the cylinder are presented. Equilibrium conditions of a deformed edge are obtained by the linear superposition of the resultant force distribution at the edge before and after deformations take place. Three scalar equilibrium conditions are thus obtained for the free edge. In addition, a fourth equation arises from the moment at the free edge.

Solutions are obtained for the four equilibrium edge conditions derived with the assumption of a semi-infinite cylinder. For the case where Poisson's ratio is 0.3, the critical stress ratio vs. the parameter which is indicative of the number of complete buckled waves around the cylinder is plotted. The lateral displacement is also plotted as a function of the axial coordinate.

The authors are at Stanford University.
8 References.

27. Newman, M. and Reiss, E. L.: Axisymmetric Snap Buckling of Conical Shells. NASA TN D-1510, December 1962, pp. 451-462.

Discussion of recent results pertaining to cone buckling limited to axisymmetric deformations is presented. The close relation of conical and spherical cap problems is discussed. Results are presented graphically for buckling of simply supported and shallow truncated cones.

Relaxation buckling behavior is discussed. Differential equations previously obtained are presented. The boundary value problems associated with simply supported and shallow truncated cones are discussed. Friedrichs' energy buckling criterion as applied to cones is used to indicate the existence of an intermediate load where the energy of deformation is the same as that for the lowest buckling load. Numerical data are presented in charts of load vs. a geometrical dimensionless number and load vs. axial shortening for specific dimensionless geometrical parameters (two sets of geometrical parameters are employed). Also shown are curves of upper, lower, and intermediate buckling loads vs. the geometric parameter employed.

Bifurcation buckling is also discussed. Upper and lower bounds for the intermediate buckling load are presented.

The authors are at Republic Aviation Corporation and New York University, respectively.
18 References.

28. North American Aviation, Inc.: Testing of Unstiffened Metal Foil Cylinders With and Without Internal Pressure. North American Aviation, Inc., Missile Development Division, Missile Test Laboratory, AL-2679, September 1957.

An extensive series of tests performed on both pressurized and unpressurized thin-gage metal monocoque cylinders loaded in axial compression, bending, torsion, and combined loading is described. Dimensions of the 171 test cylinders made of 18-8 ½H steel and 2S H18 aluminum alloy sheet with one 3/8 inch wide axial seam bonded with Epon VI adhesive cured at 200°F or seamwelded are presented in tabular form. Cylinder diameters were maintained at 17½ inches.

The test setup is described in detail. A schematic diagram of the load rig is shown. For developing testing procedures, the first cylinders were made of two-ply laminated fiberglass. Material characteristics for the fiberglass are included. Construction techniques for fiberglass cylinders were repeatedly changed but no suitable method was found. Three other materials were investigated and, unlike the fiberglass, the main problems encountered were not intrinsic in the material but were in the fabrication procedure. The three other materials tested were 18-8 ½H steel, 25 H18 aluminum alloy, and Mylar. Suitability of brazing, bonding, and seam welding were investigated. Brazing is rejected and bonding was adoped. Bonded seams separated in some cases of high torsion loading. Consequently, seam welding is adopted. A complete description of the model tests undertaken to study model characteristics is included.

A schematic diagram of the loading system for torsion and compression of the metal cylinders is included. Two loading conditions were studied in the tests. In one case, axial stress due to internal pressure was present whereas in the other case this stress was balanced by an axial strut.

Test results are presented in tabular form and are plotted as average values on a graph. The graphical data are presented in dimensionless form. In addition to failure data, skin buckling stress and torque deflection data are also presented graphically. The data plotted have no corrections for length-radius ratio but show consistent results for the materials and thicknesses tested. For all buckling load values, the buckling load is defined as the load that caused plastic buckling.

The author is not stated.
1 Reference.

29. Peterson, J. P.: Bending Tests of Ring-Stiffened Circular Cylinders. NACA TN 3735, July 1956.

A series of bending tests on ring-stiffened circular cylinders were made of 25 cylinders loaded to failure. Variations in the ratio of ring spacing to radius and in the radius to thickness ratio were incorporated in the test specimens. Variations in radius to thickness ratio were confined to values between 120 and 750. Ring spacing to radius ratios were $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, and 4 for the tests. Cylinders with diameters of 19, 30, 48, and 77 inches were tested. Rings of the 19 and 30 inch diameter cylinders were extruded Z-sections with the use of two 1x1x1/8 inch angle sections and a 3 inch sheet of 1/8 inch thickness. Rings in all cases were heavy in order to reduce the possibility of general instability failures. Specimen details are presented in tabular form. All specimens were constructed of 7075-T6 aluminum alloy.

Two test rigs were used and photographs of these rigs are presented. Loads were applied by means of a hydraulic jack. Friction in the loading was accounted for in one rig but was not established for the other frame.

Tabular data of the bending moment sustained by the specimens at buckling are presented. Bending moment sustained at buckling is also presented in graphical form plotted against parameters obtained from small-deflection theory. Data indicate that decreasing stiffener spacing yields negligible gain in strength until a value of spacing to radius ratio of $\frac{1}{2}$ is reached. Mode of failure data is also obtained. A photograph of a cylinder after failure is included.

It was determined that any size effect that exists must be small and hidden due to the scatter of data obtained for cylinders of the same size. No attempt is made to determine the extent of eccentricities in structures tested.

A graph is presented that was obtained by using small-deflection theory as a guide and fairing the lower limit of the curves to the experimental data. It is conjectured that this curve is adequate in prediction of the bending strength of ring-stiffened circular cylinders where the rings are large enough to prevent general-instability failures.

The author is at Langley Aeronautical Laboratory.
7 References.

30. Reynolds, T. E.: A Graphical Method for Determining the General-Instability Strength of Stiffened Cylindrical Shells. Navy Department, David Taylor Model Basin, Structures and Materials Laboratory, Research and Development Report 1106, September 1957.

General instability strength of externally pressurized, ring-stiffened, cylindrical shells is considered. Solutions of Kendrick (Part I, Part II, and Part III) are discussed. A short-cut method of approximating Kendrick's third solution from Kendrick's second solution is presented. Shell radius, thickness, compartment length, frame spacing, and frame size are considered as significant parameters. Length to radius ratios of from approximately 2 to 10 are considered and a wide range of the shell pressure parameters are considered.

Calculated results are compared with those of Kendrick's third solution in tabular form. In all but a few scattered points, results compare to within 10 per cent of Kendrick's theory. Calculations for externally framed steel cylinders are presented graphically. The appendix illustrates the use of the design curves by a numerical example. Also included in the appendix is an approximate formula for the calculation of the cylinder frame strength parameter.

The author is at David Taylor Model Basin. Washington, D. C.
7 References.

31. Ross, R. D.: An Electrical Computer for the Solution of Shear-lag and Bolted-Joint Problems. NACA TN 1281, May 1947.

The analog between the distribution of forces in flat stiffened panels and bolted joints and the current in a ladder type resistor network is used to obtain analogue solutions for the stresses in shear lag, box beam, and bolted joint problems.

A pilot model of a more elaborate analogue system is described. Twenty-seven 1000 ohm adjustable resistors with a sensitivity of $\pm$ 1 per cent or $\pm$ 2 ohms (whichever is least) are used. A wiring diagram and photographs of the network are included. Current is supplied by electronic regulating circuits. Current regulator diagrams are also shown. Currents are measured with a milliameter having a sensitivity of about $\frac{1}{4}$ per cent of full scale. Errors inherent in the analogue are discussed.

The analogy between the physical and analogue variables is presented in tabular form for the case of shear-lag. Boundary conditions for the physical and analogue cases are discussed. An example problem is chosen to compare the solution of the analogue with the exact solution of Kuhn. The procedure for determining the appropriate resistances in the analogue network is discussed and examples are included. The effect of the selection of bay lengths is discussed and examples for different bay lengths are compared with the exact solution graphically. The length of the bays is shown to have a pronounced effect on the maximum stress. A box beam loaded in bending is treated similarly. The cover panel of the beam is the same as the plate of the previous example.

Results are plotted graphically and are compared with the exact analysis. Both shearing and normal stresses are shown.

The analogy (for the bolted-joint problem) between the physical and analogue variables is also presented in tabular form. Plate constants are calculated and the appropriate resistances are determined. Results are tabulated and shown to compare favorably with an analytical method.

The author is at Langley Aeronautical Laboratory.
3 References.

32. Seide, P.: A Donnell-Type Theory for Asymmetrical Bending and Buckling of Thin Conical Shells. Guided Missiles Research Division. The Ramo-Wooldridge Corp., Report No. GM-TM-103, Contract No. AF (600)-1190, July 1956.

Asymmetric bending and buckling of circular cones is treated by retaining certain terms omitted in the analysis by Hoff. A figure indicating the notation used is presented.

In the case of bending, expressions for the strains and curvatures of the middle surface presented by Love are used with the modification in the curvatures of deleting the terms involving the circumferential displacement to define the stress and moment resultants. Equations of equilibrium and the boundary conditions are established by the principle of minimum potential energy. The change in potential energy due to virtual displacements is written and minimized yielding the stress and moment resultant equations in the axial, circumferential, and radial directions. The equations of equilibrium are shown to be identical to those for plane stress or strain in polar coordinates if a simple transformation is made relating the transformed polar angle to the actual polar angle and the cone semi-vertex angle. Stress and moment resultants are written in terms of a stress function and the transformed polar system. A single fourth order equation is eventually obtained in the stress function and the normal shell deflection. Expressions for the tangential and longitudinal deformations are presented.

The procedure for buckling is much the same as that for bending except for the fact that the energy expression is modified by adding the energy stored in the plate during buckling by the middle surface stress and moment resultants to that present prior to buckling. Non-linear strain expressions proposed by Langhaar are used. By methods similar to those for bending, a single eighth order equation is obtained for buckling in the variable of normal skin deformation that reduces to a fourth order equation in the special case of a concentrated load where the force resultants are given by the membrane stress solution.

In both cases (bending and buckling) the equations reduce to the Donnell equations for cylinders, the circular flat plate equations, and the axisymmetric cone equations when the appropriate cone parameter is varied. The equations are, in both cases, fourth order partial differential equations with variable coefficients that are not solvable except in special cases. It is stated that numerical solutions are apt to be useless due to the poor convergence of the series solution.

The author is at Ramo-Wooldridge Corporation, Guided Missile Research Division.
9 References.

33. Seide, P.: A Survey of Buckling Theory and Experiment for Circular Conical Shells of Constant Thickness. Aerospace Corp. El Segundo Calif., Report No. TDR-169 (3560-30) TN 1, November 1962.

Buckling theory for circular conical shells is reviewed. Small deflection theory forms the basis for the major part of the review although large deformation results are mentioned. Axisymmetric buckling as well as asymmetric cases are considered. Axial compression, combinations of axial compression and internal pressure, external hydrostatic pressure, combined axial load and external pressure, and torsion are considered and critical loads are given where results are available. Edge conditions considered are both simply supported and clamped. References for large deformation results are given.

Available experimental data are compared with theoretical results for the loading conditions mentioned above. In addition, experimental results are given for the case of pure bending, bending and internal pressure, and bending, axial compression and internal pressure for which there are no theoretical results available. All results are presented graphically. In the case of combined loading, interaction curves are presented.

Recommendations for future research are made and gaps in the current knowledge (theoretical and experimental) are indicated.

The author is at Aerospace Corporation
61 References.

34. Singer, J.: The Effect of Axial Constraint on the Instability of Thin Circular Cylindrical Shells Under External Pressure. Technion Research and Development Foundation, Israel Institute of Technology, Department of Aeronautical Engineering, Technical Note No. 1, Contract No. AF 61(052)-123, September 1959.

The effect of axial restraint on the instability of a cylindrical shell under hydrostatic pressure or uniform lateral pressure is analysed by the Rayleigh-Ritz method. Restraints are considered active from initial loading or active at the start of buckling. Restraints are applied at cylinder ends.

Axial and hoop stresses are derived for the two cases of end restraints. Expressions for stresses and displacements are substituted into the equation for total potential energy and the usual stability determinant is formed by minimizing this expression. The critical pressure thus obtained reduces to the expression obtained by Von Mises in the absence of axial restraints. The expression derived is for the case of axial restraint applied at the beginning of loading.

Instability of a cylindrical shell under uniform external lateral pressure is analysed in the same manner. An equation is presented for the critical stress which differs from the usual membrane stress by a factor α. An expression for α is presented for both the hydrostatic and uniform external lateral pressure cases. In the lateral pressure case the equation for α is equivalent to that derived by Batdorf.

The author is at Israel Institute of Technology.
9 References.

35. Smirnov, A. F.: Some New Methods for Solving Structural Problems by Means of Computers. Symposium on the Use of Computers in Civil Engineering. Laboratorio Nacional de Engenharia Civll, Paper No. 71, October 1962.

A new computational method for solving complex problems in structural mechanics is presented. The analysis is based on the notion that the highest order derivative in a differential equation may be represented by Lagrange polynomials. Hence, the vector of the highest derivative is represented by the polynomial of column and integral matrices.

Initial connecting parameters, represented by matrix equations, constitute the boundary value problem. A relationship between the lower and higher order derivatives is furnished employing the matrix of connection parameters. Forces and moments are then determined by means of a linear transformation.

In this paper the major difficulty consists of determining the composition of the integral matrix referred to above. Although this method is particularly suited for high speed digital computers, the problem is not out of range for the desk calculator in some cases. Applicability of the approach is extended to both ordinary and partial differential equations.

The author is of USSR Academy of Construction and Architecture.
No References.

36. Suer, H. S., Harris, L. A. and Shene, W. T.: The Bending Stability of Thin-Walled Unstiffened Circular Cylinders Including the Effects of Internal Pressure. North American Aviation, Inc., AL 2733, December 1957.

A statistical, semi-empirical analysis of the buckling strength of unpressurized and pressurized cylinders under axial compression previously presented by the authors is extended to the bending of unpressurized and pressurized cylindrical shells. In the analysis, a semi-empirical approach is used to account for the effect of initial imperfections. For unpressurized cylinders, theoretical parameters from the classical form of the buckling equation for long cylinders are used.

Ten tests were performed on unpressurized cylinders and 48 tests were performed on pressurized cylinders. Cylinders were fabricated from 0.0032 inch and 0.0087 inch thick, half-hard, 18-8 stainless steel foil. Overall cylinder lengths were 21.5 inches and cylinder radii were 8.75 inches. Radius to thickness ratios were approximately 1000 and 2730. Most of the cylinders were fabricated by wrapping the sheet and using Epon IV to bond 3/8 inch longitudinal seam. Other cylinders were fabricated by seam welding.

Two series of tests were performed. In one series, the longitudinal component of internal pressure was balanced by an applied axial compressive load. In the other series, the longitudinal component of internal pressure was not balanced. An internal pressure variation range of 0 to 24 psi, was used. A schematic drawing of the test jig is presented. Cylinders were positioned so that the longtudinal seam fell on the neutral axis and no appreciable difference in performance of cylinders of the two types of construction was noted. Internal pressure was pneumatically supplied. Bending and axial compressive loads were supplied by hydraulic struts and measured with SR-4 load cells. Buckling loads were visually determined by observing ripples in the cylinder surface.

Experimental results are presented in both tabular and graphical form and are compared with results of applicable previous investigations. Photographs of typical buckle patterns for both pressurized and unpressurized cylinders are presented. Experimental values of bending buckling coefficient are plotted against radius to thickness ratio and both a 90 per cent and a 99 per cent probability curve is statistically defined to obtain design data. Due to sparsity of data for radius to thickness ratios greater than 1500 additional testing is recommended to verify the shape of the design curve. Also the lower bound design curve (99 per cent probability) is suggested for pretensioned cylinders.

The authors are at North American Aviation, Inc.
14 References.

37. Tennyson, R. C.: An Experimental Investigation of the Behavior of Stiffened Plates in Axial Compression. Institute of Aerophysics, UTIA Technical Note No. 57, September 1961.

Two experimental test programs are undertaken to study the effects of curvature on panels with unstable stringers and compare them with existing data and to establish the trend of the variation of ultimate strength ratio with the stringer-to-plate stiffness ratio while keeping the stringer-to-plate area ratio constant.

The first test program consists of tests of eight panels made of 24S-T3 aluminum sheet with a thickness of 0.040 inches. Stringers, of the same sheet material with a thickness of 0.075 inches were rolled channel sections attached by means of bonding (HYSOL mixture) and round head machine screws. The panels had overall dimensions of 16 inches x 18 inches. The stringers were 1.94 inches x 0.707 inches (0.707 inches being the flange dimension). The 16 inch width of the panel was cast in Wood's metal. Each specimen had two stiffeners and three bays.

The second test program uses five flat and five curved (radius of curvature of 32 inches) panels. The same basic construction was used as in the first test program except that the stringers were made of HOMALITE photoelastic plastic with a thickness of D.1875 inches. The panel ends were not cast in Wood's metal.

In addition to the two test programs, an auxillary program is undertaken to determine the stress-strain relations of the stringers and plates. Short (4 inch) pack specimens that consisted of four channel columns were tested for each stringer material. Plate material was tested in tension.

Linear theory and Wenzek's equation are used as a theoretical guide for initial panel buckling. Also the theory of Seide-Stein is used. Experimental results show good agreement with both theoretical guides.

To determine effective widths, the stress distribution of Sechler and Dunn is assumed. This is confirmed by integrating across the panel and comparing the computed load with the actual load. The experimentally derived effective widths are compared graphically with analytical results of Sechler and Dunn, and Wenzek. It is found that the analytical results are rather conservative.

Ultimate strengths of the panel are compared with analytical results of Sechler and Dunn graphically and it is determined that the analytical results are conservative by a factor of 3 per cent.

The second program allows color photographs to be taken of the stringers which indicated that the stringer failure was by Euler-type bending.

The author is at University of Toronto.
9 References.

38. Terry, E. L. and McClaren, S. W.: Biaxial Stress and Strain Data on High Strength Alloys for Design of Pressurized Components. Chance Vought Corp., Technical Documentary Report No. ASD-TDR 62-401, July 1962.

Authors Abstract

A cross shaped specimen is developed for generating complete biaxial stress-strain curves under 1:1 and 2:1 biaxial tension stress ratio loading. Tests on several materials have shown that the specimen has good reliability.

The influence of strength level on the behavior of the 5CrMoV steel under biaxial loading is investigated. These tests show that by lowering the uniaxial strength level from 280 to 260 ksi, the shattering type failure observed at the 280 ksi level ceased to exist. However, the biaxial failure strains did not increase as the length level was decreased.

Pressure vessel tests which are conducted show that the shattering type behavior obtained from the biaxial specimens is indicative of poor resistance to crack-like flaws. Good correlation was obtained between the failure stresses from the pressure vessels and the biaxial specimens.

Notch toughness tests are conducted to obtain a correlation between these tests and the biaxial specimen tests. No correlation could be shown between the notch toughness values and the biaxial failure strains. However, the notch toughness tests corroborated the conclusion that the shattering type failure in the biaxial test is indicative of poor resistance to crack-like flaws in the material.

The biaxial stress and strain data are presented in a form which can be used directly in the design of biaxially loaded components. In addition, the test materials are ranked according to the efficiency parameters "biaxial ductility rating," "resistance to crack-like flaws" and "biaxial strength / weight."

The authors are at Chance Vought Corporation.
12 References.

39. van der Neut, Arie: General Instability of Orthogonally Stiffened Cylindrical Shells. NASA TN D-1510, December 1962, pp. 309-321.

Recent work at the National Aeronautics Research Institute

(Amsterdam) is reviewed. This work involves two structural schemes--the orthotropic shell and the shell with continuously distributed stringers and discrete rings. Only the case where the buckling wave length is of the order of twice the distance between rings is considered in the latter scheme. More recent developments are discussed that account for pressure, the correct stiffness matrix for the skin panels in post-buckling, and for stringer bending due to hoop stresses. No numerical evaluation of the stability equation with the use of the stiffness matrix is being performed.

The applicability of linearized theory in prediction of buckling loads is discussed. Conclusions are drawn that, due to the coexistance of symmetric and several asymetric modes, non-linearities must be introduced to produce better correlation between experiment and theory. Imperfections in the structure that cause eccentricities in section properties are discussed. Imperfections are confined to the class where a difference in the local radius of the cylinder and the average cylinder radius is on the order of the panel thickness. Ring imperfections can usually be kept small in comparison to ring height and it is assumed that the linear theory is adequate in the prediction of buckling loads in structures with small ring imperfections. Pressurization is found to restore, to some extent, the imperfections in secions and bring about closer agreement with linear theory.

Earlier work on orthotropic shells evolved five significant structural parameters and two mode parameters. Expressions involving the seven parameters were simplified in the earlier work for five classes of buckling modes classified by the ratio of longitudinal and circumferential wave lengths. Two cases referred to short longitudinal waves and small numbers of circumferential waves and two referred to long longitudinal waves. In the latter, no explicit formula for critical load could be obtained but a rapidly convergent numerical solution was possible. Earlier work also considered the use of external or internal stiffening. The former is considered more efficient.

Earlier work on shells with discrete rings involved the interaction of six structural parameters and two mode parameters that govern the ring displacements. Only short wave lengths are considered due to the discrete rings. Comparison of calculated buckling loads predicted by orthotropic shell theory and discrete ring theory indicates little difference in these predicted loads. This is shown graphically as a plot of the load parameter versus the number of rings per half wave length.

Recent work reviewed includes the post-buckling behavior of skin panels, load pressure difference, and hoop stresses that induce stringer bending.

Post-buckling behavior is discussed. The stiffness matrix is presented and modifications of the prebuckling matrix to obtain the post-buckling matrix are discussed. The use of flat panel data in curved panel calculations is discussed and it is conjectured that flat panel data is applicable in general instability curved panel analysis if allowances are made for initial curvature.

Reduction of lateral panel stiffnesses due to hoop stresses not balanced by pressure difference is discussed. Differences in stringer and ring deflection due to stringer bending is considered and it is proposed that future research include a method by which spring stiffness in a mechanical model of a structure, where rings and stringers are

SATURN IB

EFFECT OF CENTER OF PRESSURE LOCATION ON LATERAL DISPLACEMENT

ONE RETRO-ROCKET FAILURE

NOTE: 3 DEG INITIAL ANGLE OF ATTACK

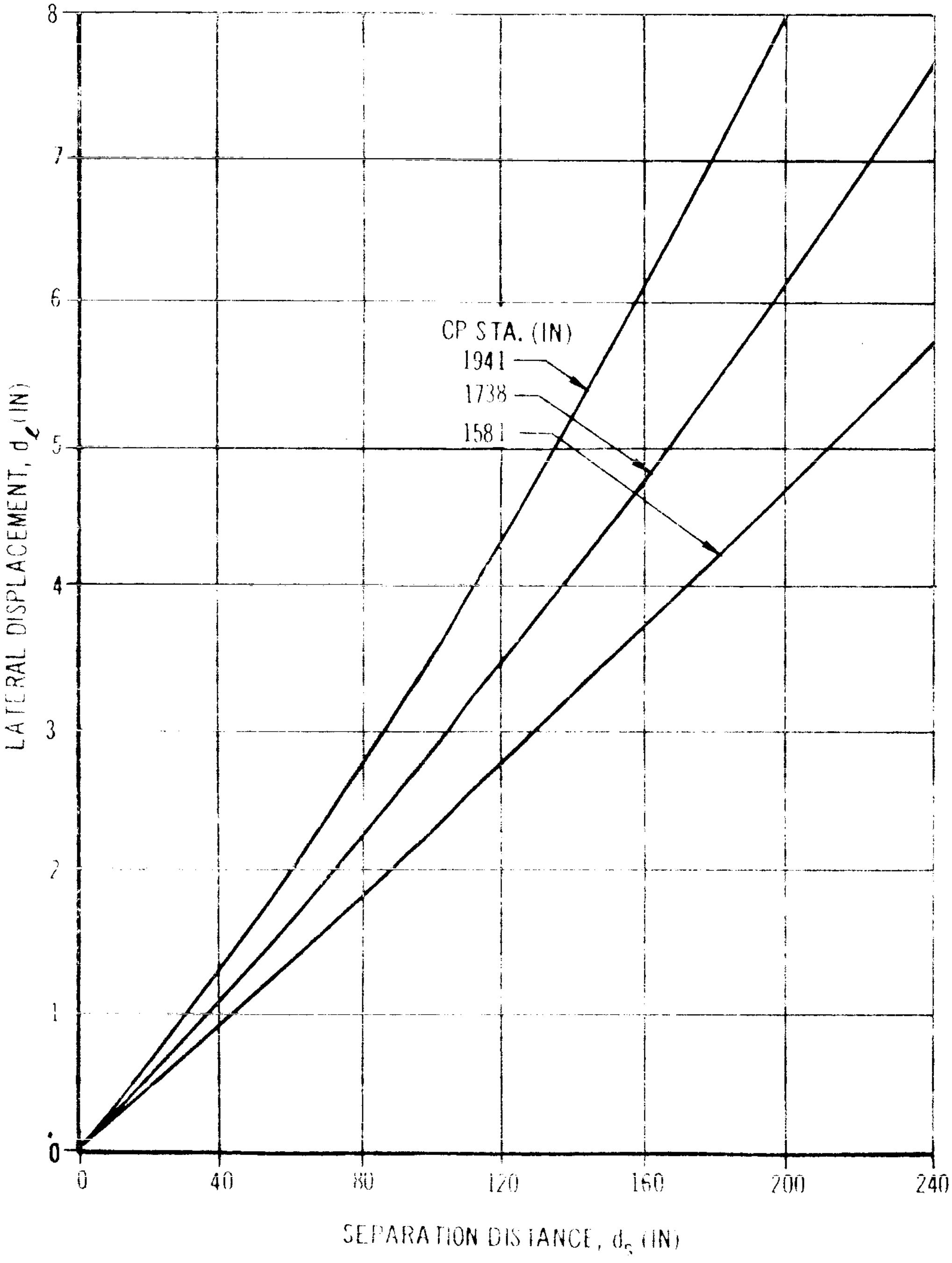

FIGURE 22

connected with springs, can be determined as a function of wave length to ring pitch ratio.

The author is at Technological University. Delft (Netherlands).
5 References.

APPENDIX E

THEORETICAL ANALYSIS OF THE STATIC GENERAL INSTABILITY OF AN ORTHTROPIC CIRCULAR CYLINDER SUBJECTED TO AN AXIAL LOAD, END MOMENT AND UNIFORM RADIAL PRESSURE

Prepared by

William S. Viall

Carl C. Steyer

The work reported in this appendix was supported jointly by NASA Contract NAS 8-5168 and NASA research grant NsG-381 and was also published by the University of Alabama Research Institute as Research Report No. 11.

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THEORETICAL ANALYSIS OF THE STATIC GENERAL INSTABILITY OF AN ORTHOTROPIC CIRCULAR CYLINDER SUBJECTED TO AN AXIAL LOAD, END MOMENT, AND UNIFORM RADIAL PRESSURE

by

William S. Viall[+]

Carl C. Steyer[++]

I. SCOPE

This analysis obtains numerical results using a digital computer program for the general instability eigenvalue problem that is presented for the dependent buckling load condition at any combination of the independent loading and geometry. It is not intended that this analysis be experimentally verified as a part of this investigation.

II. INTRODUCTION

Missile tank design is subjected to two design criteria; the material strength for all possible maximum load conditions, and the structural stability at these possible maximum load conditions as well as intermediate loads. The analysis of this report is limited to the stability criteria of missile design.

A missile tank is loaded with different combinations of axial load, end moment, and radial pressure. The axial load, either compressive or tensile, is a constant force per unit cross-sectional area and is colinear with the generating element of the tank. The end moment is a varying force per unit cross-sectional area and is colinear with the generating element of the tank. This force varies linearly with the distance between a diameter that is normal to the plane of the moment, and the element of cross-sectional area. The radial pressure is internal or external, depending upon the sign given to the pressure difference.

The positive directions of axial load, end moment, and radial pressure are as shown in Figure 2 and are chosen to induce tension on the element of the tank at the origin of the axes, Figure 1.

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Functions performed by missile tanks require that they be stiffened axially and circumferentially with internal baffles, internal and external stiffeners, bulkheads, etc. These baffles and stiffeners are integral parts of the missile tank and it becomes possible to analyze the tank as an orthotropic circular cylindrical shell. The orthotropic circular cylindrical shell is called circular shell or shell in the remainder of this report.

The loads applied to the shell are not functions of time, therefore, the investigation is limited to the static case, and the dynamic case is neglected.

The technique used in this investigation of the cylindrical shell parallels the work of Bodner (1)*, in that the general instability differential equation of equilibrium developed is a Donnell type differential equation and is obtained by the application of variational methods to the expression for total change in energy during buckling.

Results can be obtained from the Donnell Type differential equation by any one of several different methods: Ritz Method, Fourier Series Method, Galerkin Method, Method of Frobenius, etc.; all of which will yield a satisfactory solution. The Ritz Method is used in this investigation. The results obtained by the Ritz Method are as accurate as the assumed deflection expression and the solution becomes an exact solution when the deflection expression takes the form of an infinite Fourier series. The Ritz Method is mathematically the simplest of the methods mentioned above and it is readily programmed for digital computer applications.

The points of stability for the total change in energy expression of a conservative system are defined by the law of minimum potential energy when the variational principal is applied to the total change in energy. The first variation of the total change in energy is equated to zero, thereby obtaining the intrinsic boundary conditions and the equilibrium equations of the system. The Donnell Type differential equation is obtained by applying a differential operator to the equilibrium equations of the system. An assumed deflection expression is substituted into the Donnell equation and the resulting residual force equation is minimized for each of the unknown constants of the deflection expression. This minimization yields a system of homogeneous simultaneous equations, and the stability determinant of these equations is solved for the independent variable.

---

*Indicates reference number - see Appendix B.

The law of minimum potential energy requires that the second variation of the total change in energy expression of the system be positive for the points of stable equilibrium and negative for points of unstable equilibrium. The second variation is not performed due to the anticipated difficulty of the mathematics, and the minimum positive value of the independent variable is assumed as the point of stable equilibrium.

Experimental evidence obtained by Harris, Suer, Skene and Benjamin (2) indicates that isotropic circular shell test specimens subjected to axial load with and without radial pressure fail somewhere between the stable and unstable equilibrium points and that the failure point is primarily dependent upon the quality of the specimen. The almost perfect specimens fail at the point approaching the point of unstable equilibrium. As the imperfections of the specimens become larger or more numerous the specimen fails at a point closer to the point of stable equilibrium. Theories developed for unpressurized cylinders with axial loads by von Kármán and Tsien (3), Leggett and Jones (4), and Tsien (5) using the large deflection theory have attempted to explain the deviation between theoretical and experimental results. These theories are still considered as inadequate since their results cannot be readily adapted as design criteria. A similar approach was used for pressurized cylinders with axial loads by Donnell and Wan (6) with more success, but a deviation still exists.

Small deflection theory of shell analysis states that all terms greater than second degree in the total change in energy expression may be neglected. The small deflection theory is used in this investigation and allows the development of the Donnell Type linear differential equation which can be readily solved for a certain particular type of loading. That the small deflection theory is applicable to certain shell configurations is questioned by some investigators as indicated above. The answer to this question is left for further analytical work associated with the experimental evaluation of this investigation.

The classical small deflection theory for isotropic shell stability is limited to the range R/h-values** less than 200. Some missile tanks have R/h-values of 1000 and there have been indications that this value may reach 2000. This indicates that the R/h-values for orthotropic shells that represent stiffened shells should be modified, or the valid range

---

** See Appendix A for list of symbols and definitions.

of the theory extended for orthotropic shells. Arguments for a modification of R/h-values using a modified h-value, which we will call (h_{eq}), are based on the dependence of the stability criterion on the bending rigidity of the shell. Similar arguments are used for the extensional stiffness. Suggested values for (h_{eq}) are:

$$h_{eq} = (12\ I_{eq})^{1/3}$$

where I_{eq} is the composite moment of inertia of the shell plus stiffeners;

$$h_{eq} = [6(I_{xx} + I_{ss})]^{1/3}$$

where I_{xx} and I_{ss} are the equivalent composite moments of inertia in the axial and circumferential directions, respectively; or $h_{eq} = f(r)$, where r is the radius of gyration of a unit element of the orthotropic shell. It is believed that if the shell is analyzed with $R/h_{eq} = 200$ the small deflection theory will be applicable.

Investigation is being conducted (7) which may allow the proper selection of an R/h-value for orthotropic shells with an h_{eq}-value. Again the question of an R/h_{eq}-value for orthotropic shells is left for experimental evaluation and/or results of investigations in progress.

For short cylinders ($R \geq L$) the assumed deflection expression, Equation 26, reduces to the Euler column expression when the cylinder is simply supported, if the circumferential deflection terms become constant. For long cylinders ($R \leq L/3$) the buckling becomes independent of the boundary conditions. In these ranges this analysis is valid for values of $\pi R/L$, but the intermediate range ($L \leq R \leq L/3$) the results should again be experimentally verified.

III. ASSUMPTIONS

The following assumptions are made in this analysis of circular shells.

1. The shell is composed of linearly elastic material.

2. The stiffeners and baffles are integral parts of the shell, thus creating an orthotropic shell, and the unstiffened and unbaffled shell reduces to the isotropic case.

3. The shell stresses in the unbuckled but stressed state are determined by elementary beam theory.

4. The strain equations, Equation 1, are similar to those used by Bodner (1) except for certain second degree terms. The second degree terms are included in this analysis since their effect although unknown, is considered significant. The last term on the right hand side of the e_{ss} equation, Equation 1, includes a non-dimensional constant k. With values of k equal to one and zero, the effect of this term on the final results can be determined.

5. The work in the circumferential direction is neglected in the determination of the total change in energy expression. This is based on the symmetry of both of the σ_{ss} stresses and the cylindrical shell geometry.

6. The pre-buckling deformation discussed by Donnell and Wan (6) and Stein (8) are neglected. The effect of these deformations should be investigated during experimental verification.

7. In the development of the Donnell equation, all terms above the second order in the total energy expression are discarded. Neglecting the terms above second order simplifies the mathematics and insures a small deflection theory approach to the analysis.

8. Localized or panel instability is neglected in this analysis and only the general instability of the shell is investigated.

9. The assumed deflection expression, Equation 26, contains only 6 circumferential terms, but can be extended to any number desired. The use of six terms requires that a cubic equation be solved and this solution can easily be programmed. Additional terms in the deflection expression would complicate the computer program and possibly produce a computer overload. The axial term in the deflection expression contair- a nondimensional constant m which is used to obtain any number of buckling modes.

IV. CYLINDRICAL SHELL GEOMETRY AND STRESS-STRAIN RELATIONS

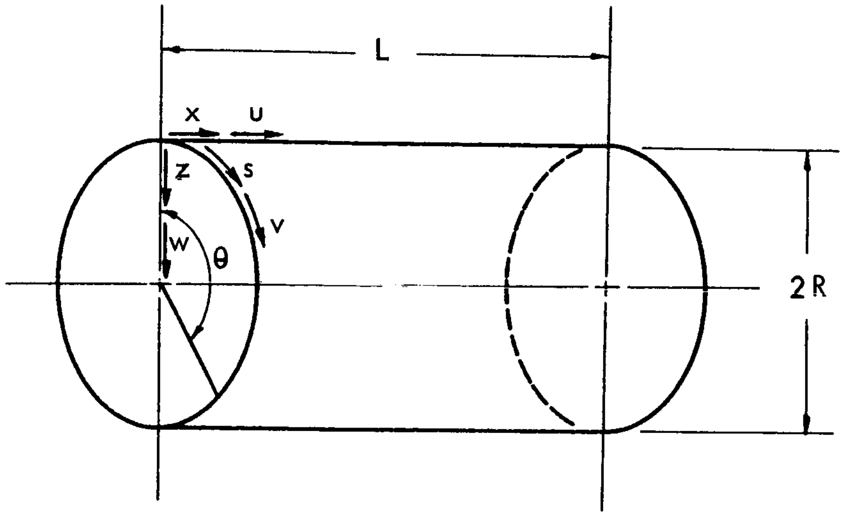

Fig. 1: Coordinate System and Displacements of the Circular Cylindrical Shell.

The coordinate system and corresponding middle-surface displacements for the circular cylindrical shell are shown in Figure 1.

The expressions used for the buckling strains in the shell wall; written in terms of the shell middle-surface displacements, u, v, and w; are the same as those given in Reference (1) with some additional terms, and are written as follows:

$$\begin{aligned} e_{xx} &= u,_{x} + (1/2)\, w^{2},_{x} - z\, w,_{xx} \\ e_{ss} &= v,_{s} - (w/R) + (1/2)(w,_{s} + v/R)^{2} - z(w,_{ss} + kw/R^{2}) \\ e_{xs} &= (1/2)\,[u,_{s} + v,_{x} + w,_{x}\,(w,_{s} + v/R)] - z(w,_{xs} + v,_{x}/2R) \end{aligned} \qquad (1)$$

where e_{xx}, e_{ss}, and e_{xs}, are the axial, circumferential, and shear strains, respectively, that occur during the buckling process; R is the radius of the cylinder; and k is a nondimensional constant. When the subscript or subscripts associated with the middle-surface displacements are preceeded by a comma, they denote differentiation with respect to the indicated succeeding coordinate variables.

The stress-strain relationships for a homogenous orthotropic material in generalized plane stress, as given by Reference (1), can be written as follows:

$$
\begin{aligned}
\sigma_{xx} &= E_x (e_{xx} + \mu_{sx} e_{ss})/(1 - \mu_{sx}\mu_{xs}) \\
\sigma_{ss} &= E_s (e_{ss} + \mu_{xs} e_{xx})/(1 - \mu_{sx}\mu_{xs}) \\
\sigma_{xs} &= G e_{xs}
\end{aligned}
\tag{2}
$$

where σ_{xx}, σ_{ss}, and σ_{xs}, are the axial, circumferential, and shear stresses, respectively; E_x and E_s are the moduli of elasticity averaged over the shell thickness in the axial and circumferential directions, respectively; G is the average shear modulus; and μ_{xs} and μ_{sx} are Poisson's ratios from the x to the s and s to the x directions, respectively.

For convenience in later calculations certain constants, similar to those given in Reference (1), are introduced and are written as follows:

$$
\begin{aligned}
\alpha_1 &= E_x h/2(1 - \mu_{xs}\mu_{sx}) \\
\alpha_2 &= E_s h/2(1 - \mu_{xs}\mu_{sx}) \\
\alpha_3 &= Gh/8 \\
D_1 &= E_x h^3/24(1 - \mu_{xs}\mu_{sx}) \\
D_2 &= E_s h^3/24(1 - \mu_{xs}\mu_{sx}) \\
D_3 &= Gh^3/96
\end{aligned}
\tag{3}
$$

where h is the shell thickness; the α's correspond to the extensional stiffness of the shell; and the D's correspond to the bending rigidity of the shell.

The following relationship between the elastic constants, based on Maxwell's reciprocal theorem, is noted for later use.

$$E_s \mu_{xs} = E_x \mu_{sx} \tag{4}$$

V. CYLINDRICAL SHELL LOADING AND STRESS RESULTANTS

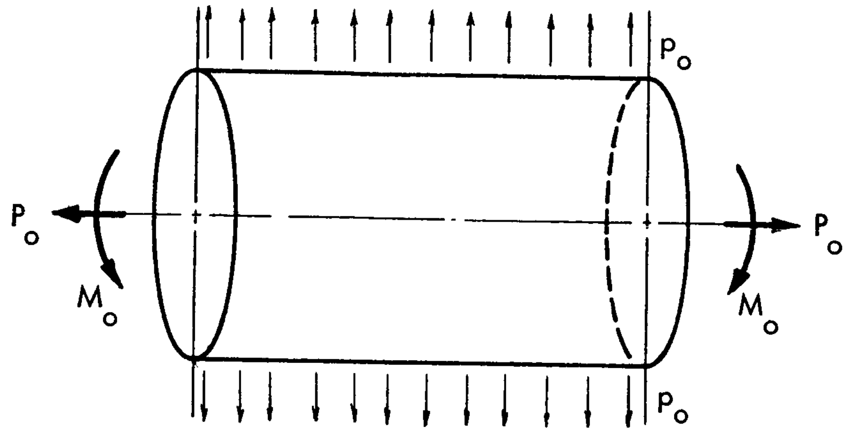

Fig. 2: Loading of the Circular Cylindrical Shell

The positive shell loads p_o, P_o, and M_o are shown with respect to the coordinate system; where p_o is the uniform radial pressure, P_o is the resultant axial load, and M_o is the resultant end moment. The positive loads are directed in order that positive stresses (tensil) are induced at the origin of the coordinate system.

The following stress resultants are defined:

$$\overline{N}_{xx} = \int_{-h/2}^{h/2} \bar{\sigma}_{xx} dz \qquad \overline{N}_{ss} = \int_{-h/2}^{h/2} \bar{\sigma}_{ss} dz \qquad \overline{N}_{xs} = \int_{-h/2}^{h/2} \bar{\sigma}_{xs} dz \tag{5}$$

where $\overline{N}_{xx}$, $\overline{N}_{ss}$, and $\overline{N}_{xs}$ are the axial, circumferential, and shear stress resultants in the shell wall, respectively, prior to buckling; and $\bar{\sigma}_{xx}$, $\bar{\sigma}_{ss}$, and $\bar{\sigma}_{xs}$ are the axial, circumferential, and shear stresses in the shell wall, respectively, prior to buckling. In general the barred symbols indicate stresses, strains, and stress resultants in the shell prior to buckling, while un-barred symbols indicate stresses and strains that occur in the shell during the buckling process.

According to elementary beam and shell theory, the shell loading will induce the following stresses in the shell wall.

$$\bar{\sigma}_{xx} = (1/h)\,[M_a \cos(s/R) + (R/2)(P_a + p_o)]$$

$$\bar{\sigma}_{ss} = (1/h)\,p_o R \qquad (6)$$

$$\bar{\sigma}_{xs} = 0$$

where $M_a = M_o/\pi R^2$ and $P_a = P_o/\pi R^2$.

Substituting Equation (6) into Equation (5) and integrating over the shell thickness the stress resultants become:

$$\overline{N}_{ss} = p_o R \qquad (7)$$

$$\overline{N}_{xx} = M_a \cos(s/R) + (R/2)(p_o + P_a)$$

VI. STRAIN ENERGY, POTENTIAL ENERGY, TOTAL CHANGE IN ENERGY, AND VARIATION IN TOTAL CHANGE IN ENERGY EXPRESSIONS.

The instability differential equations of equilibrium will be derived using a procedure similar to that given in Reference (1). The criterion of buckling for an elastic system is that the potential energy of the system is a minimum. Stated mathematically, the variation of the change in energy of the system due to buckling, with respect to the displacements, must be zero; or:

$$\delta\,(U + V) = 0 \qquad (8)$$

where U is the change in strain energy of the shell during buckling, V is the change in potential energy of the applied loads during buckling, and δ indicates a variation of the sum with respect to displacements.

The change in the strain energy of the shell is given by the following expression:

$$U = \int_{V_s} [(\bar{\sigma}_{xx} e_{xx} + \bar{\sigma}_{ss} e_{ss} + \bar{\sigma}_{xs} e_{xs}) + (1/2)(\sigma_{xx} e_{xx} + \sigma_{ss} e_{ss} + \sigma_{xs} e_{xs})]\, dV_s \qquad (9)$$

where $\bar{\sigma}_{xx}$, $\bar{\sigma}_{ss}$, and $\bar{\sigma}_{xs}$ are the membrane stresses in the shell wall in the stressed but unbuckled state and they are assumed to be constant during buckling; σ_{xx}, σ_{ss}, and σ_{xs} are the superimposed buckling stresses; e_{xx}, e_{ss}, and e_{xs} are the buckling strains; and V_s is the volume of the shell wall.

The change in potential energy of the shell during buckling has components in the z and x directions and the total change in potential energy during buckling is given by the following expression:

$$V = -\int_{A_s} p_o(-w)dA_s \; - \int_{A_e} \bar{\sigma}_{xx} u \, dA_e = \int_{A_s} p_o w dA_s - \int_{A_s} h \bar{\sigma}_{xx} u_{,x} dA_s \tag{10}$$

where A_s and A_e are the surface and cross sectional areas of the shell, respectively; and u is given by the expression $u = \int_0^L u_{,x} \, dx$.

The total change in energy, strain energy plus potential energy, during buckling is obtained by adding Equation (9) to Equation (10); substituting Equations (1), (2), (4), (5), (6), and (7) into Equations (9) and (10); and integrating over the shell thickness. The expression for total change in energy is given by the following expression:

$$
\begin{aligned}
U + V = \int_{A_s} & \{p_o [v_{,s} R + (w_{,s}^2 R/2) + w_{,s} v + (v^2/2R) + (w_{,x}^2 R/4)] \\
& + P_a [w^2_{,x} R/4] + M_a \cos(s/R) [w_{,x}^2/2]\} \, dA_s \\
+ \int_{A_s} & \{[E_x h/2(1-\mu_{xs}\mu_{sx})][u^2_{,x}] + [E_s h/2(1-\mu_{xs}\mu_{sx})][v^2_{,s} + (w^2/R^2) \\
& - (2v_{,s} w/R) + \mu_{xs} u_{,x} v_{,s} - (\mu_{xs} u_{,x} w/R)] \\
& + [Gh/8][v^2_{,x} + u^2_{,s} + w_{,x}^2 w_{,s}^2 + (w_{,x}^2 v^2/R^2) + 2v_{,x} u_{,s} + 2v_{,x} w_{,x} w_{,s} \\
& + (2v_{,x} w_{,x} v/R) + 2u_{,s} w_{,x} w_{,s} + (2u_{,s} w_{,x} v/R) + (2w_{,x}^2 w_{,s} v/R)]\} \, dA_s \\
+ \int_{A_s} & \{[h^3 E_x/24(1-\mu_{sx}\mu_{xs})][w_{,xx}^2 + \mu_{sx} w_{,xx} w_{,ss} + (k \mu_{sx} w_{,xx} w/R^2)] \\
& + [h^3 E_s/24(1-\mu_{sx}\mu_{xs})][w_{,ss}^2 + (2w_{,ss} k w/R^2) + (k^2 w^2/R^4) \\
& + \mu_{xs} w_{,xx} w_{,ss} + (\mu_{xs} k w_{,xx} w/R^2)] \\
& + [Gh^3/96][4w_{,xs}^2 + (v_{,x}^2/R^2) + (4w_{,xs} v_{,x}/R)]\} dA_s
\end{aligned}
\tag{11}
$$

Substituting Equation (3) into Equation (11) and discarding all third order and greater terms the expression for total change in energy reduces to:

$$
\begin{aligned}
U + V = \int_{A_s} \{ & p_o [v,_s R + (w,_s^2 R/2) + w,_s v + (v^2/2R) + (w,_x^2 R/4)] \\
& + P_a [w,_x^2 R/4] + M_a \cos(s/R) [w,_x^2/2] \\
& + \alpha_1 [u,_x^2] + \alpha_2 [v,_s^2 + (w^2/R^2) - (2v,_s w/R) + \mu_{xs} u,_x v,_s - (\mu_{xs} u,_x w/R)] \\
& + \alpha_3 [v,_x^2 + u,_s^2 + 2v,_x u,_s] + D_1 [w,_{xx}^2 + \mu_{sx} w,_{xx} w,_{ss} + (k \mu_{sx} w,_{xx} w/R^2)] \\
& + D_2 [w,_{ss}^2 + (2w,_{ss} kw/R^2) + (k^2 w^2/R^4) + \mu_{xs} w,_{xx} w,_{ss} + (\mu_{xs} kw,_{xx} w/R^2)] \\
& + D_3 [4w,_{xs}^2 + (v,_x^2/R^2) + (4w,_{xs} v,_x/R)] \} \, dA_s
\end{aligned}
\tag{12}
$$

where α_1, α_2, α_3, D_1, D_2, and D_3 are as defined in Equation (3). Several authors, including the author of Reference (1) have proven that the ommitted terms are negligible. The use of terms up to the second order will result in a linear differential equation.

Applying the variational principal,

$$
\delta F = \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' + \frac{\partial F}{\partial y''} \delta y'' + \ldots\ldots\ldots + \frac{\partial F}{\partial y^n} \delta y^n,
\tag{13}
$$

to Equation (12), the following expression for the variation in the total change in energy with respect to the displacements u, v, and w is obtained:

$$
\begin{aligned}
\delta(U + V) = \int_{A_s} \Big[& \{ \alpha_2 [(2w/R^2) - (2v,_s/R) - (\mu_{xs} u,_x/R)] \\
& + D_2 [(2w,_{ss} k/R^2) + (2\mu_{xs} w,_{xx} k/R^2) + (2k^2 w/R^4)] \} \delta w \\
& + \{ M_a w,_x \cos(s/R) + (p_o R w,_x/2) + (P_a R w,_x/2) \} \delta w,_x \\
& + \{ p_o (w,_s R + v) \} \delta w,_s \\
& + \{ D_1 [2w,_{xx} + 2\mu_{sx} w,_{ss} + (2\mu_{sx} kw/R^2)] \} \delta w,_{xx}
\end{aligned}
\tag{14}
$$

$$+ \{D_2[2w,_{ss} + (2kw/R^2) + 2\mu_{xs}w,_{xx}]\}\delta w,_{ss}$$

$$+ \{D_3[8w,_{xs} + (4v,_x/R)]\}\delta w,_{xs}$$

$$+ \{a_1[2u,_x] + a_2[\mu_{xs}v,_s - (\mu_{xs}w/R)]\}\delta u,_x$$

$$+ \{a_3[2u,_s + 2v,_x]\}\delta u,_s + \{p_o[(v/R) + w,_s]\}\delta v \qquad \text{(14 Con't.)}$$

$$+ \{a_3[2v,_x + 2v,_s] + D_3[(2v,_x/R^2) + (4w,_{xs}/R)]\}\delta v,_x$$

$$+ \{p_oR + a_2[2v,_s - (2w/R) + \mu_{xs}u,_x]\}\delta v,_s \Big| dA_s$$

Equation (14) is simplified further by setting $dA_s = dxds$; applying the following identity from calculus of variations, $\delta \frac{dx}{dy} = \frac{d}{dy}(\delta_x)$; and integrating between the limits of 0 to L for dx, and 0 to $2\pi R$ for ds. The final form of the expression for the variation in the total change in energy of the orthotropic cylindrical shell during the buckling process is as follows:

$$\begin{aligned}
\delta(U+V) = \int_0^L\int_0^{2\pi R} \Big[& \{2a_2[(w/R^2) - (v,_s/R) - (\mu_{xs}u,_x/2R)] \\
& + 2D_2[(2w,_{ss}k/R^2) + (\mu_{xs}w,_{xx}k/R^2) + (k^2w/R^4) + w,_{ssss} + \mu_{xs}w,_{xxss}] \\
& - [M_aw,_{xx}\cos(s/R) + (Rw,_{xx}/2)(p_o + P_a)] - [p_o(w,_{ss}R + v,_s)] \\
& + 2D_1[w,_{xxxx} + \mu_{sx}w,_{xxss} + (\mu_{sx}kw,_{xx}/R^2)] \\
& + 4D_3[2w,_{xxss} + (v,_{xxs}/R)]\}\delta w \\
& + \{-2a_1[u,_{xx} + (\mu_{sx}v,_{xs}/2) - (\mu_{sx}w,_x/2R)] - 2a_3[u,_{ss} + v,_{xs}]\}\delta u \\
& + \{-2a_2[v,_{ss} - (w,_s/R) + (\mu_{xs}u,_{xs}/2)] + [p_o(w,_s + \tfrac{v}{R})] \\
& - 2a_3[v,_{xx} + u,_{xs}] - 4D_3[(v,_{xx}/2R^2) + (w,_{xxs}/R)]\}\delta v \Big] dxds
\end{aligned} \tag{15}$$

$$+ \int_o^L \Big[\{[p_o(w,_s R + v)] - 2D_2[w,_{sss} + \mu_{xs} w,_{xxs} + (kw,_s/R^2)]$$

$$-4D_3[2w,_{xxs} + (v,_{xx}/R)]\}_o^{2\pi R} \ \delta w$$

$$+\{2a_3[u,_s + v,_x]\}_o^{2\pi R} \delta u$$

$$+\{p_o R + 2a_2[v,_s - (w/R) + (\mu_{xs} u,_x/2)]\}_o^{2\pi R} \delta v$$

$$+\{2D_2[w,_{ss} + \mu_{xs} w,_{xx} + (kw/R^2)]\}_o^{2\pi R} \delta w,_s \Big] \ dx$$

$$+ \int_o^{2\pi R} \Big[\{[m_a w,_x \cos(s/R) + (Rw,_x/2)(p_o + P_a)]$$

$$-2D_1[w,_{xxx} + \mu_{sx} w,_{xss} + (\mu_{sx} k w,_x/R^2)] \qquad \text{(15 Con't.)}$$

$$-4D_3[2w,_{xss} + (v,_{xs}/R)]\}_o^L \ \delta w$$

$$+\{2a_1[u,_x] + 2a_2[(\mu_{xs} v,_s/2) - (\mu_{xs} w/2R)]\}_o^L \ \delta u$$

$$+\{2a_3[v,_x + u,_s] + 4D_3[(v,_x/2R^2) + (w,_{xs}/R)]\}_o^L \delta v$$

$$+\{2D_1[w,_{xx} + \mu_{sx} w,_{ss} + (\mu_{sx} k w/R^2)]\}_o^L \delta w,_x \Big] \ ds$$

$$+ \Big[\{4D_3[2w,_{xs} + (v,_x/R)]\}_o^L \Big]_o^{2\pi R} \ \delta w$$

VII. EQUILIBRIUM EQUATIONS AND NATURAL BOUNDARY CONDITIONS

The variation in the total change in energy of the system must vanish for any of the arbitrary virtual displacements δu, δv, and δw when the system is in equilibrium. When Equation (15) is equated to zero the integrands of the surface integral must vanish, since the virtual displacements are arbitrary, and the following stability equilibrium equations are obtained:

$$u,_{xs}[-a_3 - (a_2 \mu_{xs}/2)] + v[p_o/2R] + v,_{xx}[(-D_3/R^2) - a_3] + v,_{ss}[-a_2]$$
$$+ w,_{xxs}[-2D_3/R] + w,_s[(p_o/2) - (a_2/R)] = 0 \qquad (16)$$

$$v,_{xs}[-a_3-(a_2\mu_{xs}/2)] + u,_{xx}[-a_1] + u,_{ss}[-a_3] + w,_x[a_2\mu_{xs}/2R] = 0 \tag{17}$$

$$\begin{aligned} & u,_x[-a_2\mu_{xs}/2R] + v,_s[(-a_2/R) + (-p_o/2)] + v,_{xxs}[2D_3/R] \\ & + w[(a_2/R^2) + (k^2D_2/R^4)] + w,_{xx}[(2D_1\mu_{sx}k/R^2) - (m_a/2)\cos(s/R) - (R/4)(p_o + P_a)] \\ & + w,_{ss}[(2D_2k/R^2) - (p_oR/2)] + w,_{xxxx}[D_1] + w,_{ssss}[D_2] \\ & + w,_{xxss}[2\mu_{sx}D_1 + 4D_3] = 0 \end{aligned} \tag{18}$$

The following natural boundary conditions are obtained from Equation (15) when the constant term and the integrands of the line integrals vanish for any arbitrary virtual displacement, and derivative of an arbitrary virtual displacement.

$$\begin{aligned} & \Big[\, [w,_{xs} + (v,_x/2R)]_o^L \,\Big]_o^{2\pi R} = 0 \\ & [w,_{xx} + \mu_{sx}w,_{ss} + (k\mu_{sx}w/R^2)]_o^L = 0 \\ & [w,_{ss} + \mu_{xs}w,_{xx} + kw/R^2]_o^{2\pi R} = 0 \\ & [u,_s + v,_x]_o^{2\pi R} = 0 \\ & [2u,_x + \mu_{sx}v,_s - (\mu_{sx}w/R)]_o^L = 0 \\ & [p_oR + 2a_2(v,_s - w/R + \mu_{xs}u,_x/2)]_o^{2\pi R} = 0 \\ & [a_3(v,_x + u,_s) + 2D_3(v,_x/2R^2 + w,_{xs}/R)]_o^L = 0 \\ & [p_o(w,_sR + v) - 2D_2(w,_{sss} + \mu_{xs}w,_{xxs} + kw,_s/R^2) \\ & \qquad - 4D_3(2w,_{xxs} + v,_{xx}/R)]_o^{2\pi R} = 0 \\ & [M_aw,_x\cos(s/R) + (Rw,_x/2)(p_o + P_a) - 2D_1(w,_{xxx} + \mu_{sx}w,_{xss} + \mu_{sx}kw,_x/R^2) \\ & \qquad - 4D_3(2w,_{xss} + v,_{xs}/R)]_o^L = 0 \end{aligned} \tag{19}$$

VIII. DEVELOPMENT OF A DONNELL TYPE DIFFERENTIAL EQUATION FOR THE STATIC CASE

The stability equilibrium equations are written in the following form:

$$u,_{xs} = ap_1 v + a_4 v,_{xx} + a_2 v,_{ss} + a_5 w,_{xxs} + ap_2 w,_s \tag{16a}$$

$$v,_{xs} = a_1 u,_{xx} + a_3 u,_{ss} + a_6 w,_x \tag{17a}$$

$$\begin{aligned} c_1 u,_x + cp_1 v,_s + c_2 v,_{xxs} = b_4 w + [bp_2 - M_a R\cos(s/R)]w,_{xx} + bp_1 w,_{ss} \\ + b_1 w,_{xxxx} + b_3 w,_{xxss} + b_2 w,_{ssss} \end{aligned} \tag{18a}$$

where:

$$\begin{aligned} ap_1 &= p_o a_7 \\ ap_2 &= p_o a_8 + a_9 \\ bp_1 &= p_o b_5 + b_6 \\ bp_2 &= p_o b_7 + P_a b_7 + b_8 \\ cp_1 &= p_o c_3 + c_4 \end{aligned} \tag{20}$$

and:

$$\begin{aligned} a_1 &= -2\alpha_1/(\mu_{xs}\alpha_2 + 2\alpha_3) \\ a_2 &= -2\alpha_2/(\mu_{xs}\alpha_2 + 2\alpha_3) \\ a_3 &= -2\alpha_3/(\mu_{xs}\alpha_2 + 2\alpha_3) \\ a_4 &= -2(D_3 + \alpha_3 R^2)/R^2(\mu_{xs}\alpha_2 + 2\alpha_3) \\ a_5 &= -4D_3/R(\mu_{xs}\alpha_2 + 2\alpha_3) \\ a_6 &= \alpha_2\mu_{xs}/R(\mu_{xs}\alpha_2 + 2\alpha_3) \\ a_7 &= 1/R(\mu_{xs}\alpha_2 + 2\alpha_3) \\ a_8 &= 1/(\mu_{xs}\alpha_2 + 2\alpha_3) \end{aligned} \tag{20 a}$$

$a_9 = -2a_2/R(\mu_{xs} a_2 + 2a_3)$

$b_1 = 2D_1 R$

$b_2 = 2D_2 R$

$b_3 = 4D_1 R \mu_{sx} + 8D_3 R$

$b_4 = 2(R^2 a_2 + k^2 D_2)/R^3$

$b_5 = -R^2$

$b_6 = 4kD_2/R$ (20a Con't.)

$b_7 = -R^2/2$

$b_8 = 4\mu_{sx} kD_1/R$

$c_1 = a_2 \mu_{xs}$

$c_2 = -4D_3$

$c_3 = R$

$c_4 = 2a_2$

A linear differential operation is defined as follows:

$$Q = a_1 ap_1 \frac{\partial^2}{\partial x^2} + a_3 ap_1 \frac{\partial^2}{\partial s^2} + a_1 a_4 \frac{\partial^4}{\partial x^4} + (a_1 a_2 + a_3 a_4 - 1) \frac{\partial^4}{\partial x^2 \partial s^2} + a_2 a_3 \frac{\partial^4}{\partial s^4} \qquad (21)$$

By successive differentiation and combination, Equations (16a) and (17a) will have the following form:

$$Qu = - [a_6 ap_1 w,_x + (a_2 a_6 + ap_2) w,_{xss} + a_4 a_6 w,_{xxx} + a_5 w,_{xxxss}] \qquad (22a)$$

$$Q_v = - [(a_1 ap_2 + a_6) w,_{xxs} + a_3 ap_2 w,_{sss} + a_1 a_5 w,_{xxxxs} + a_3 a_5 w,_{xxsss}] \qquad (22b)$$

Operating on Equating (18a) with the differential operator defined in Equation (21) results in the following:

$$Q(c_1 u,_x + cp_1 v,_s + c_2 v,_{xxs})$$

$$= Q \; [b_4 w + bp_2 w,_{xx} - M_a R \cos(s/R) w,_{xx} + bp_1 w,_{ss} + b_1 w,_{xxxx} + b_3 w,_{xxss} + b_2 w,_{ssss}] \qquad (18b)$$

By utilizing Equations (22a) and (22b), all the u and v terms in Equation (18b) can be eliminated. The resulting equation, an eighth order differential equation in w, is the required Donnell-type differential equation and is given as follows:

$$
\begin{aligned}
& h_{80} w_{,xxxxxxxx} + h_{62} w_{,xxxxxxss} + h_{44} w_{,xxxxssss} + h_{26} w_{,xxssssss} \\
& + h_{08} w_{,ssssssss} + [h_{60} + h_{c60} M_a \cos(s/R)]\, w_{,xxxxxx} + [h_{42} + h_{c42} M_a \cos(s/R)]\, w_{,xxxxss} \\
& + [h_{24} + h_{c24} M_a \cos(s/R)]\, w_{,xxssss} + h_{06} w_{,ssssss} + h_{s41} M_a \sin(s/R)\, w_{,xxxxs} \\
& + h_{s23} M_a \sin(s/R)\, w_{,xxsss} + [h_{40} + h_{c40} M_a \cos(s/R)]\, w_{,xxxx} + [h_{22} + h_{c22} M_a \cos(s/R)]\, w_{,xxss} \\
& + h_{04} w_{,ssss} + h_{s21} M_a \sin(s/R)\, w_{,xxs} + [h_{20} + h_{c20} M_a \cos(s/R)]\, w_{,xx} + h_{02} w_{,ss} = 0
\end{aligned}
\tag{23}
$$

where:

$$
\begin{aligned}
h_{80} &= d_{80} \\
h_{62} &= d_{62} \\
h_{44} &= d_{44} \\
h_{26} &= d_{26} \\
h_{08} &= d_{08} \\
h_{60} &= d_{60} + e_{60} p_o + f_{60} P_a \\
h_{c60} &= g_{60} \\
h_{42} &= d_{42} + e_{42} p_o + f_{42} P_a \\
h_{c42} &= g_{42} \\
h_{24} &= d_{24} + e_{24} p_o + f_{24} P_a \\
h_{c24} &= g_{24}
\end{aligned}
\tag{24}
$$

$$h_{06} = d_{06} + e_{06}p_o$$

$$h_{s41} = \bar{g}_{41}$$

$$h_{s23} = \bar{g}_{23}$$

$$h_{40} = d_{40} + e_{40}p_o + \bar{e}_{40}p_o^2 + ef_{40}p_oP_a$$

$$h_{c40} = g_{40} + eg_{40}p_o$$

$$h_{22} = d_{22} + e_{22}\,p_o + \bar{e}_{22}p_o^2 + ef_{22}\,p_oP_a \qquad \text{(24 Con't.)}$$

$$h_{c22} = eg_{22}\,p_o + g_{22}$$

$$h_{04} = d_{04} + e_{04}\,p_o + \bar{e}_{04}\,p_o^2$$

$$h_{s21} = e\bar{g}_{21}\,p_o + \bar{g}_{21}$$

$$h_{20} = e_{20}\,p_o$$

$$h_{c20} = eg_{20}p_o + g_{20}$$

$$h_{02} = e_{02}p_o$$

and:

$$d_{80} = a_1a_4b_1$$

$$d_{62} = a_1a_5c_2 + b_1(a_1a_2 + a_3a_4 - 1) + a_1a_4b_3$$

$$d_{44} = a_3a_5c_2 + a_2a_3b_1 + b_3(a_1a_2 + a_3a_4 - 1) + a_1a_4b_2 \qquad (25)$$

$$d_{26} = a_2a_3b_3 + b_2(a_1a_2 + a_3a_4 - 1)$$

$$d_{08} = a_2a_3b_2$$

$$d_{60} = a_1a_4b_8$$

$$e_{60} = a_1a_7b_1 + a_1a_4b_7$$

$$f_{60} = a_1a_4b_7$$

$$g_{60} = -Ra_1a_4$$

$$d_{42} = a_5c_1 + a_1a_5c_4 + a_1a_9c_2 + a_6c_2 + b_8(a_1a_2 + a_3a_4 - 1) + a_1a_4b_6$$

$$e_{42} = a_1a_5c_3 + a_1a_8c_2 + a_1a_4b_5 + a_3a_7b_1 + a_1a_7b_3 + b_7(a_1a_2 + a_3a_4 - 1)$$

$$f_{42} = b_7(a_1a_2 + a_3a_4 - 1)$$

$$g_{42} = -R(a_1a_2 + a_3a_4 - 1)$$

$$d_{24} = a_3a_5c_4 + a_3a_9c_2 + a_2a_3b_8 + b_6(a_1a_2 + a_3a_4 - 1)$$

$$e_{24} = a_3a_5c_3 + a_3a_8c_2 + b_5(a_1a_2 + a_3a_4 - 1) + a_3a_7b_3 + a_1a_7b_2 + a_2a_3b_7$$

$$f_{24} = a_2a_3b_7 \qquad \text{(25 Con't.)}$$

$$g_{24} = -Ra_2a_3$$

$$d_{06} = a_2a_3b_6$$

$$e_{06} = a_2a_3b_5 + a_3a_7b_2$$

$$\bar{g}_{41} = 2(a_1a_2 + a_3a_4 - 1)$$

$$\bar{g}_{23} = 4a_2a_3$$

$$d_{40} = a_4a_6c_1 + a_1a_4b_4$$

$$e_{40} = a_1a_7b_8$$

$$ef_{40} = a_1a_7b_7$$

$$g_{40} = (1/R)(a_1a_2 + a_3a_4 - 1)$$

$$eg_{40} = -Ra_1a_7$$

$$\bar{e}_{40} = a_1a_7b_7$$

$$d_{22} = a_2a_6c_1 + a_1a_9c_4 + a_6c_4 + b_4(a_1a_2 + a_3a_4 - 1) + a_9c_1$$

$$e_{22} = a_8c_1 + a_1a_9c_3 + a_1a_8c_4 + a_6c_3 + a_3a_7b_8 + a_1a_7b_6$$

$$\bar{e}_{22} = a_1a_8c_3 + a_1a_7b_5 + a_3a_7b_7$$

$$ef_{22} = a_3a_7b_7$$

$$g_{22} = (1/R)(6a_2a_3)$$

$$eg_{22} = a_3a_7R$$

(25 Con't.)

$$d_{04} = a_3a_9c_4 + a_2a_3b_4$$

$$e_{04} = a_3a_9c_3 + a_3a_8c_4 + a_3a_7b_6$$

$$\bar{e}_{04} = a_3a_8c_3 + a_3a_7b_5$$

$$e\bar{g}_{21} = 2a_3a_7$$

$$\bar{g}_{21} = (1/R^2)(-4a_2a_3)$$

$$e_{20} = a_6a_7c_1 + a_1a_7b_4$$

$$eg_{20} = (1/R)\, a_3a_7$$

$$g_{20} = (1/R^3)(-a_2a_3)$$

$$e_{02} = a_3a_7b_4$$

IX. DETERMINATION OF THE CRITICAL RESULTANT END MOMENT BY USE OF THE RITZ METHOD

The classical Ritz method solution for the static buckling of a cylindrical shell, as shown in Reference (1), requires that an assumption be made for the shape of the buckled cylinder. The following expression is assumed for the radial deflection:

$$w = [A_1 \cos(s/R) + A_2 \cos(2s/R) + A_3 \cos(3s/R) + A_4 \cos(4s/R) + A_5 \cos(5s/R) + A_6 \cos(6s/R)]\ [\sin(m\pi x/L)] \tag{26}$$

where A_1 through A_6 are arbitrary displacement parameters; and m is an arbitrary positive interger representing the number of buckling modes in the axial direction. The assumed radial deflection expression satisfies the boundary condition for the coordinates x and s. These boundary conditions, for a simply supported shell, are zero deflection and moment at the ends of the cylinder and a periodicity of 2π, respectively, for the x and s coordinates. The boundary conditions represented mathematically are:

$$w(x,s) = w(0,s) = w(L,s) = 0 \tag{27a}$$

$$w,_{xx}(x,s) = w,_{xx}(0,s) = w,_{xx}(L,s) = 0 \tag{27b}$$

for the x coordinate, and:

$$w(x,s) = w(x,s + 2\pi) \tag{27c}$$

for the s coordinate.

For convenience, Equation (26) will be written in the following summation form:

$$w = [A_n \cos(ns/R)]\ [\sin(m\pi x/L)] \tag{26a}$$

where the n, an interger, is the summing index and has the values 1 through 6.

Substitution of Equation (26a) into Equation (23) will result in a residual force per unit area F, and Equation (23) can be written in the following form:

$$F = [\sin(m\pi x/L)]\ [G_1 + G_2 + G_3] \tag{28}$$

where

$$G_1 = M_a A_n B_{1n} \cos(s/R) \cos(ns/R)$$

$$G_2 = M_a A_n B_{2n} \sin(s/R) \sin(ns/R) \qquad (29)$$

$$G_3 = A_n B_{3n} \cos(ns/R)$$

and

$$B_{1n} = R^3 [- h_{c60} (\lambda^6/R^6) - h_{c42} (\lambda^4 n^2/R^6) - h_{c24} (\lambda^2 n^4/R^6) + h_{c40} (\lambda^4/R^4) + h_{c22} (\lambda^2 n^2/R^4) - h_{c20} (\lambda^2/R^2)]$$

$$B_{2n} = R^3 [- h_{s41} (\lambda^4 n/R^5) - h_{s23} (\lambda^2 n^3/R^5) + h_{s21} (\lambda^2 n/R^3)] \qquad (30)$$

$$B_{3n} = R^3 [h_{80} (\lambda^8/R^8) + h_{62} (\lambda^6 n^2/R^8) + h_{44} (\lambda^4 n^4/R^8) + h_{26} (\lambda^2 n^6/R^8) + h_{08} (n^8/R^8) - h_{60} (\lambda^6/R^6) - h_{42} (\lambda^4 n^2/R^6) - h_{24} (\lambda^2 n^4/R^6) - h_{06} (n^6/R^6) + h_{40} (\lambda^4/R^4) + h_{22} (\lambda^2 n^2/R^4) + h_{04} (n^4/R^4) - h_{20} (\lambda^2/R^2) - h_{02} (n^2/R^2)]$$

$$\lambda = m\pi R/L$$

Equation (26a) can be written again with a summation notation but using a different index, in the following form:

$$w = [A_r \cos(rs/R)] [\sin(m\pi x/L)] \qquad (26b)$$

where the r, an interger, is the summing index and has the values 1 through 6. Equation (26b) can now be written in the form:

$$w = G_o \sin(m\pi x/L) \qquad (31)$$

where

$$G_o = A_r \cos(rs/R) \qquad (32)$$

The work done by the residual force, F, during the radial deflection of buckling is obtained from the product of Equations (28) and (31) and is given as follows:

$$Fw = [\sin^2(m\pi x/L)]\ [G_oG_1 + G_oG_2 + G_oG_3] \tag{33}$$

The expression for Fw must be evaluated over the surface area of the cylindrical shell to obtain the total work W expression given as follows:

$$W = \int_0^L \int_0^{2\pi R} Fw\ ds\ dx \tag{34}$$

Substitution of Equation (33) into (34) will result in the following expression:

$$W = \int_0^L \int_0^{2\pi R} [\sin^2(m\pi x/L)]\ [G_oG_1 + G_oG_2 + G_oG_3]\ ds\ dx \tag{35}$$

Integrating Equation(35)with respect to x will give the following:

$$W = (L/2) \int_0^{2\pi R} [G_oG_1 + G_oG_2 + G_oG_3]\ ds \tag{36}$$

Evaluation of the integrals of the products G_oG_1, G_oG_2, and G_oG_3 will give the following:

$$(L/2) \int_0^{2\pi R} G_oG_1\ ds = M_a B_{1n} A_n A_r (L/2) \int_0^{2\pi R} \cos(s/R) \cos(ns/R) \cos(rs/R)\ ds$$

$$= M_a B_{1n} A_n A_r (\pi RL/4), \text{ when } r = n \pm 1 \tag{37a}$$

$$(L/2) \int_0^{2\pi R} G_oG_2\ ds = M_a B_{2n} A_n A_r (L/2) \int_0^{2\pi R} \sin(s/R)(ns/R) \cos(rs/R)\ ds$$

$$= M_a B_{2n} A_n A_r (\pi RL/4), \text{ when } r = n - 1; \tag{37b}$$

$$\text{and } = -M_a B_{2n} A_n A_r (\pi RL/4), \text{ when } r = n+1$$

$$(L/2) \int_0^{2\pi R} G_o G_3 \, ds = B_{3n} A_n A_r (L/2) \int_0^{2\pi R} \cos (ns/R) \cos (rs/R) \, ds$$

$$= B_{3n} A_n A_r (\pi RL/4) \text{ when } r = n \tag{37c}$$

All other combinations of r and n values not specified by the r–n condition equations will cause the integrals to vanish. Substitution of the specified values for r and n into Equations (37a), (37b) and (37c); evaluation of these equations; and separation of terms will result in the following general expression for the total work during the buckling due to the radial deflection:

$$W = (\pi RL/4) \left[\sum_{n=1}^{6} A_n^2 B_{3n} + M_a \sum_{n=1}^{5} A_n A_{n+1} (B_{1,n} + B_{1,n+1} - B_{2,n} + B_{2,n+1}) \right] \tag{38}$$

where the values of n are as specified on the summation symbols.

Equation (38) is minimized with respect to the arbitrary displacement parameters A_n when n again has the interger values 1 through 6. This procedure will result in the following system of algebraic equations:

$$\frac{\partial W}{\partial A_1} = 0 : \quad A_1 \bar{B}_{11} + A_2^{M_a} \bar{B}_{12} = 0 \tag{39}$$

$$\frac{\partial W}{\partial A_2} = 0 : \quad A_1^{M_a} \bar{B}_{21} + A_2 \bar{B}_{22} + A_3^{M_a} \bar{B}_{23} = 0 \tag{40}$$

$$\frac{\partial W}{\partial A_3} = 0 : \quad A_2^{M_a} \bar{B}_{32} + A_3 \bar{B}_{33} + A_4^{M_a} \bar{B}_{34} = 0 \tag{41}$$

$$\frac{\partial W}{\partial A_4} = 0 : \quad A_3^{M_a} \bar{B}_{43} + A_4 \bar{B}_{44} + A_5^{M_a} \bar{B}_{45} = 0 \tag{42}$$

$$\frac{\partial W}{\partial A_5} = 0 : \quad A_4^{M_a} \bar{B}_{54} + A_5 \bar{B}_{55} + A_6^{M_a} \bar{B}_{56} = 0 \tag{43}$$

$$\frac{\partial W}{\partial A_6} = 0 : \quad A_5^{M_a} \bar{B}_{65} + A_6 \bar{B}_{66} = 0 \tag{44}$$

where:

$$\bar{B}_{n,n} = 2B_{3n}; \qquad n = 1 \text{ to } 6 \tag{45}$$

$$\bar{B}_{n,n+1} = \bar{B}_{n+1,n} = (B_{1,n} + B_{1,n+1} - B_{2,n} + B_{2,n+1}); \quad n = 1 \text{ to } 5$$

The coefficients of the A_n terms in Equations (39) through (44), when written in determinate form, result in the following expression:

$$(\bar{D}) = \begin{vmatrix} \bar{B}_{11} & M_a \bar{B}_{12} & 0 & 0 & 0 & 0 \\ M_a \bar{B}_{21} & \bar{B}_{22} & M_a \bar{B}_{23} & 0 & 0 & 0 \\ 0 & M_a \bar{B}_{32} & \bar{B}_{33} & M_a \bar{B}_{34} & 0 & 0 \\ 0 & 0 & M_a \bar{B}_{43} & \bar{B}_{44} & M_a \bar{B}_{45} & 0 \\ 0 & 0 & 0 & M_a \bar{B}_{54} & \bar{B}_{55} & M_a \bar{B}_{56} \\ 0 & 0 & 0 & 0 & M_a \bar{B}_{65} & \bar{B}_{66} \end{vmatrix} \tag{46}$$

and Equations (39) through (44) can be written in the following matrix form:

$$[\bar{D}] \quad [A_n] = 0 \tag{47}$$

Since the arbitrary displacement parameters, A_n, are real; the determinant, $(\bar{D})$, must vanish for all values of A_n. Therefore, evaluation of the determinant $(\bar{D})$, which results in a sixth degree equation in M_a, will give the critical resultant moment, M_a cr, of the circular cylinder for the particular values of p_o and P_a used in the evaluation of the h-constants in Equation (24). The desired value of M_a cr is the lowest, positive, real root of the following characteristic equation:

$$T_o + T_1 M_a^2 + T_2 M_a^4 + T_3 M_a^6 = 0 \tag{48}$$

where:

$$T_o = [\bar{B}_{11}\,\bar{B}_{22}\,\bar{B}_{33}\,\bar{B}_{44}\,\bar{B}_{55}\,\bar{B}_{66}]$$

$$\begin{aligned} T_1 = -[&\bar{B}_{11}\,\bar{B}_{22}\,\bar{B}_{33}\,\bar{B}_{44}\,\bar{B}_{56}^2 + \bar{B}_{11}\,\bar{B}_{22}\,\bar{B}_{33}\,\bar{B}_{66}\,\bar{B}_{45}^2 \\ &+ \bar{B}_{11}\,\bar{B}_{22}\,\bar{B}_{55}\,\bar{B}_{66}\,\bar{B}_{34}^2 + \bar{B}_{11}\,\bar{B}_{44}\,\bar{B}_{55}\,\bar{B}_{66}\,\bar{B}_{23}^2 \\ &+ \bar{B}_{33}\,\bar{B}_{44}\,\bar{B}_{55}\,\bar{B}_{66}\,\bar{B}_{12}^2] \\ T_2 = [&\bar{B}_{11}\,\bar{B}_{22}\,\bar{B}_{34}^2\,\bar{B}_{56}^2 + \bar{B}_{11}\,\bar{B}_{44}\,\bar{B}_{23}^2\,\bar{B}_{56}^2 + \bar{B}_{11}\,\bar{B}_{66}\,\bar{B}_{23}^2\bar{B}_{45}^2 \\ &+ \bar{B}_{33}\,\bar{B}_{44}\,\bar{B}_{12}^2\,\bar{B}_{56}^2 + \bar{B}_{33}\,\bar{B}_{66}\,\bar{B}_{12}^2\,\bar{B}_{45}^2 + \bar{B}_{55}\,\bar{B}_{66}\,\bar{B}_{12}^2\,\bar{B}_{34}^2] \\ T_3 = -[&\bar{B}_{12}^2\,\bar{B}_{34}^2\,\bar{B}_{56}^2] \end{aligned} \tag{49}$$

X. CONCLUSIONS

The general instability of an orthotropic circular cylinder subjected to an axial load, end moment, and uniform radial pressure has been analyzed by a technique paralleling the technique used by Bodner (1). The analysis has been successfully programmed, see Appendix C, and the program has been run with arbitrary data. The results obtained with the arbitrary data could only be visually checked and were within the range of expected results. The program has not been used in conjunction with experimental investigations.

XI. RECOMMENDATIONS

It is assumed that this investigation of orthotropic shells will be continued on an experimental basis, and that the experiments will attempt to verify and/or modify the existing analysis as well as refine and modify the computer program that has been written. The recommendations that are stated are intended as a guide for the experimental investigators.

The deflection expression, Equation 26, should be extended to a minimum of 12 circumferential deflection terms and possibly extended to 16 or 24 terms should computer capacity allow this extension. This extension will improve the accuracy of the analysis.

The axial term of the deflection expression, the sine term, should be extended to contain a cosine term, that is, $\sin(m\pi x/L) + \cos(m\pi x/L)$. The axial term will then allow a variation of end conditions, which become significant in the short cylinder range and possibly the intermediate cylinder range. This modified axial term can also be used to induce deflections due to the pre-buckling stresses.

An additional term can be added to the deflection expression to account for the initial imperfections of the cylinder.

The discarded roots of the characteristic equation should be mathematically investigated, and the meaning of the imaginary roots should be ascertained.

The sensitivity of the program should be checked for each of the dependant variables, geometric and loading. Each modification of the program should be checked for the possible changes in sensitivity that can be expected.

A normalization of the final program is recommended which will allow a comparison with other information existing in the field.

Since stability of orthotropic shells is both a general and local stability problem the program can be extended to include the local stability problem by evaluating existing investigations in this field.

Results obtained by other investigators can be checked with the program to determine whether or not the program is valid.

APPENDIX A

SYMBOL TABLE

A_e	- Cross-sectional area of the shell.
A_s	- Surface area of the middle surface of the shell.
A_1, A_2, etc.	- Arbitrary displacement parameters for the assumed deflection expression.
a_1, a_2, etc.	- Constants for the stability equilibrium equation defined by Equation 20a.
ap_1, ap_2, etc.	- Constants for the stability equilibrium equation defined by Equation 20.
B_{1n}, B_{2n}, B_{3n}	- Generalized constants defined by Equation 30.
$\overline{B}_{n,n}$, $\overline{B}_{n,n+1}$	- Generalized constants for the stability determinant defined by Equation 45.
b_1, b_2, etc.	- Constants for the stability equilibrium equation defined by Equation 20a.
bp_1, bp_2, etc.	- Constants for the stability equilibrium equation defined by Equation 20.
c_1, c_2, etc.	- Constants for the stability equilibrium equation defined by Equation 20a.
cp_1, cp_2, etc.	- Constants for the stability equilibrium equation defined by Equation 20.
D_1, D_2, D_3	- Bending rigidities for the axial, circumferential, and shear strains respectively.
$(\overline{D})$	- Stability determinant.
d_{80}, d_{60}, etc.	- Constants for the Donnell differential equation defined by Equation 25.
E	- Modulus of elasticity for the isotropic case.
E_x, E_s	- Moduli of elasticity averaged over the axial and circumferential directions, respectively.

e_{xx}, e_{ss}, e_{xs}	- Axial, circumferential, and shear strains, respectively, occurring during the buckling process, defined by Equation 1.
e_{60}, e_{42}, etc.	- Constants for the Donnell differential equation defined by Equation 25.
$\bar{e}_{40}$, $\bar{e}_{22}$, $\bar{e}_{04}$	- Constants for the Donnell differential equation defined by Equation 25.
ef_{40}, ef_{22}	- Constants for the Donnell differential equation defined by Equation 25.
eg_{40}, eg_{22}, eg_{20}	- Constants for the Donnell differential equation defined by Equation 25.
$e\bar{g}_{21}$	- Constants for the Donnell differential equation defined by Equation 25.
F	- Residual force per unit area remaining in the shell as a result of the assumed deflection expression.
f_{60}, f_{42}, f_{24}	- Constants for the Donnell differential equation defined by Equation 25.
G	- Average shear modulus, where $G = E/(1 + \mu)$
G_0	- Constant defined by Equation 32.
G_1, G_2, G_3	- Constants for the residual force equation defined by Equation 29.
g_{60}, g_{42}, etc.	- Constants for the Donnell differential equation defined by Equation 25.
$\bar{g}_{41}$, $\bar{g}_{23}$, $\bar{g}_{21}$	- Constants for the Donnell differential equation defined by Equation 25.
h	- Shell wall thickness.
h_{eq}	- Shell wall thickness modified for the orthotropic case.
h_{80}, h_{60}, etc.	- Constants for the Donnell differential equation defined by Equation 24.
h_{c60}, h_{c42}, etc.	- Constants for the Donnell differential equation defined by Equation 24.

h_{s41}, h_{s23}, etc.	- Constants for the Donnell differential equation defined by Equation 24.
k	- Non-dimensional integer constant.
L	- Length of cylindrical shell.
M_a	- Modified end moment defined by $M_a = M_o/\pi R^2$
M_o	- Applied end moment.
m	- Number of buckling modes in the axial direction.
$\overline{N}_{xx}$, $\overline{N}_{ss}$, $\overline{N}_{xs}$	- Axial, circumferential, and shear stress resultants in the shell just prior to buckling defined by Equation 5.
P_a	- Modified axial load defined by $P_a = P_o/\pi R^2$.
P_o	- Applied axial load.
p_o	- Applied uniform radial pressure.
$\pi R/L$	- Circular shell radius to length ratio.
Q	- Linear differential operator defined by Equation 21.
R	- Radius of circular shell.
R/h	- Circular shell radius to thickness ratio.
s	- Circumferential coordinate of circular shell.
T_o, T_1, etc.	- Constants for the characteristic equation defined by Equation 49.
U	- Change in strain energy during buckling.
u	- Axial deformation of an element of the circular shell.
V	- Change in potential energy during buckling.
V_s	- Volume of circular shell wall.
v	- Circumferential deformation of an element of the circular shell.

W	- Total work due to the residual force during buckling.
w	- Radial defomation of an element of the circular shell.
x	- Axial coordinate of the circular shell.
z	- Radial coordinate of the circular shell.
$\alpha_1, \alpha_2, \alpha_3$	- Extensional stiffnesses for the axial, circumferential, and shear strains respectively.
δ	- Variational symbol.
θ	- Coordinate angle corresponding to the circumferential coordinate, where $\theta = s/R$.
λ	- Constant defined by Equation 30.
μ	- Poison's ratio for the isotropic case.
μ_{xs}, μ_{sx}	- Poison's ratios from the x to s and the s to x directions, respectively.
$\sigma_{xx}, \sigma_{ss}, \sigma_{xs}$	- Axial, circumferential, and shear stresses, respectively, occurring during the buckling process.
$\bar{\sigma}_{xx}, \bar{\sigma}_{ss}, \bar{\sigma}_{xs}$	- Axial, circumferential, and shear stresses, respectively, in the circular shell just prior to buckling.

APPENDIX B

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APPENDIX C

COMPUTER PROGRAM

The solution of the problem being investigated here requires that an n-th degree polynomial in M_a be solved for the lowest, positive, real or zero root. The degree of this polynomial, the characteristic equation of the stability determinant, Equation 48, is equal to the maximum value used for n in the deflection expression, Equation 26a. The applied end moment can be plus or minus and still have the same stability condition, therefore the characteristic equation can be considered as a polynomial in M_a^2 and the roots of the characteristic equation are determined by the cubic formula. The values of M_a are then obtained by taking the square root of the M_a^2 value.

In the development of this problem, the total change in energy expression, Equation 11, is manipulated by certain mathematical operations. After each manipulation a new set of constants is obtained. These new constants are defined in terms of previously defined constants, etc., and finally all constants are defined in terms of the extensional stiffnesses and bending rigidities, Equation 3, and other input variables. Therefore, the problem that the computer program must solve is an evaluation of successive sets of constants, and the solution of the characteristic equation for the desired root. A computer program type-out is shown in Appendix D, and this program is written in Fortran II for an IBM 1620 computer.

In the investigation of an orthotropic shell, the a_1, a_2, D_1 and D_2 values, Equation 3, are calculated for a particular orthotropic shell using h_{eq}. These values are then rationed to the respective isotropic shell values, a and D, which are obtained by using h values. These ratios are used as input variables in the form: A1A, A2A, D1D, and D2D; where A1A = a_1/a, etc. Similarly the input values of L and h appear in the computer program as ratios in the form $\pi R/L$ and R/h, respectively.

In any stability problem it is necessary that the sensitivity of any or all variables be investigated, and that a study of the output variable M_a for certain ranges of the input variables be made. An iterative process that incruments the input variables between certain desired limits permits these studies. All input variables can be iterated with the exception of E, μ_{xs}, μ_{sx} and R.

The iterative process requires three input values for each of the following input terms: A1A, A2A, D1D, D2D, R/h, πR/L, p_o, P_o, k and m. These values are: the initial value, also the minimum; the maximum value; and the increment by which the input variable varies between the initial and maximum values.

The program output is M_a vs. P_o. A sample output format is shown in Appendix E. This sample output format is for arbitrary values of the input variables.

When a constant value, non-incrumented value, of an input variable is used in a particular computer run, the initial value and maximum value must be the same, and the incrument should be an arbitrary positive number.

An increase in the number of terms in the deflection expression will require a change in the root solving portion of the program, since the cubic formula will no longer provide a valid solution to the characteristic equation.

The symbols used in the computer program are self-explanatory except symbols B11, B12, and B13 which are the b_1, b_2 and b_3 constants of Equation 20a, respectively. A partial list of definitions and computer program symbols is given in Appendix F.

Certain constants used in the text of this paper do not appear in the computer program. These constants have been incorporated into succeeding constants with the intent of conserving computer storage.

The program must be precompiled with format, since an overload condition exists on a 40K bit storage when the program is precompiled without format.

APPENDIX D

COMPUTER PROGRAM TYPE-OUT

```
C       PROGRAM FOR THE STABILITY ANALYSIS OF AN ORTHOTROPIC CIRCULAR
C       SHELL WITH AXIAL LOAD, END MOMENT, AND RADIAL PRESSURE.
C       THIS PROGRAM IS WRITTEN IN FORTRAN II FOR AN IBM 1620 COMPUTER.
C       INPUT DATA (5 CARDS) - ALL DATA IN 8 DIGIT FIELDS
  100   READ501,E,VXS,VSX,R
C       VALUES FOR PRESSURE AND AXIAL LOAD
        READ501,APO,POINC,POMAX,ABGPO,BGPOI,BGPOM
C       INITIAL VALUES (MINIMUM)
        READ501,AROH,ARPL,AA1A,AA2A,AD1D,AD2D,AEK,AEM
C       INCRUMENT VALUES
        READ501,ROHIN,RPLIN,A1AIN,A2AIN,D1DIN,D2DIN,EKINC,EMINC
C       MAXIMUM VALUES
        READ501,ROHMX,RPLMX,A1AMX,A2AMX,D1DMX,D2DMX,EKMAX,EMMAX
        DIMENSION V(8),     U(8),S(6,8),B1(8),B2(8),B3(8),X(6),Y(5)
C       REPEATING CONSTANTS
        PI=3.141593
        F1=1.
        F2=2.
        F3=3.
        F4=4.
        PAINC=BGPOI/(PI*R*R)
        PAMAX=BGPOM/(PI*R**2)
C       INITIALIZING STATEMENT
        EK=AEK
C       INITIALIZING STATEMENT
   10   EM=AEM
        U(1)=F1/R
        DO 222 N=2,8
  222   U(N)=U(1)**N
         DO 225 I=1,6
         DO 225 J=1,8
        D=I
  225   S(I,J)=D**J
C       INITIALIZING STATEMENT
   15   RPL=ARPL
C       OUTPUT STATEMENT
        PUNCH510,E,VXS,VSX,R,EK,EM
   25   EL=R*PI/RPL
        V(1)=EM*PI*R/EL
        DO 223 N=2,8
  223   V(N)=V(1)**N
        ROH=AROH
C       OUTPUT STATEMENT
        PUNCH512,RPL
C       INITIALIZING STATEMENT
   35   D2D=AD2D
C       OUTPUT STATEMENT
        PUNCH515,ROH
C       INITIALIZING STATEMENT
   45   D1D=AD1D
C       INITIALIZING STATEMENT
```

```
   50 A2A=AA2A
C        INITIALIZING STATEMENT
   55 A1A=AA1A
C        INITIALIZING STATEMENT
      PO=APO
C     OUTPUT STATEMENT
   60 PUNCH511,A1A,A2A,D1D,D2D
      H=R/ROH
C     CONSTANTS FOR EQULIBRIUM EQUATIONS
      AX=F1-VXS*VSX
      Q1=F1/(E*H*(VSX/(F2*AX)-F1/(F4*(F1-VXS))))
      A1=-F2*Q1*A1A*E*H/(F2*AX)
      A2=A1*A2A/A1A
      A3=-F2*Q1*E*H/(8.*(F1+VXS))
      A5=-F4*(Q1/R)*E*H**F3/(96.*(F1+VXS))
      A4=A3+A5/(F2*R)
      A6=A2*VXS/(-F2*R)
      A7=Q1/R
      A9=-F2*A6/VXS
      B11=F2*R*D1D*E*H**F3/(24.*AX)
      B12=B11*D2D/D1D
      B13=E*H**F3*(F2*D1D*VSX/AX+F1/(F1+VSX))*R/12.
      B4=-A2*R/Q1+B12*EK*EK/R**F4
      B5=-R*R
      B6=B12*F2*EK/(R*R)
      B7=-(R**F2)/F2
      B8=B11*F2*VSX*EK/(R*R)
      C1=VSX*A2A*E*H/(F2*AX)
      C =-E*H**F3/(24.*(F1+VXS))
      C4=F2*C1/VXS
C     CONSTANTS FOR DONNELL EQUATION (EQUA. 24 AND 25)
      H80=A1*A4*B11
      Q2=A1*A2+A3*A4-F1
      H62=A1*(A5*C2+A4*B13)+B11*Q2
      H44=A3*(A5*C2+A2*B11)+B13*Q2+A1*A4*B12
      H26=A2*A3*B13+B12*Q2
      H08=A2*A3*B12
      E60=A1*(A7*B11+A4*B7)
      HC60=-R*A1*A4
      D42=A1*(A5*C4+A9*C2+A4*B6)+A5*C1+A6*C2+B8*Q2
      E42=A1*(A5*R+Q1*C2+A4*B5+A7*B13)+A3*A7*B11+B7*Q2
      HC42=-R*Q2
      D24=A3*(A5*C4+A9*C2+A2*B8)+B6*Q2
      E24=A3*(A5*R+Q1*C2+A7*B13+A2*B7)+B5*Q2+A1*A7*B12
      HC24=-R*A2*A3
      E06=A3*(A2*B5+A7*B12)
      HS41=F2*Q2
      HS23=F4*A2*A3
      D40=A4*(A6*C1+A1*B4)
      D22=C1*(A2*A6+A9)+C4*(A1*A9+A6)+B4*Q2
      E22=A1*(A9*R+Q1*C4+A7*B6)+A6*R+Q1*C1+A3*A7*B8
      EB22=A1*(Q1*R+A7*B5)+A3*A7*B7
      D04=A3*(A9*C4+A2*B4)
      E04=A3*(A9*R+Q1*C4+A7*B6)
```

```
      EB04=A3*(Q1*R+A7*B5)
      H20=PO*A7*(A6*C1+A1*B4)
C     OUTPUT STATEMENT
  600 PUNCH514,PO
C     INITIALIZING STATEMENT
      BIGPO=ABGPO
      PA=BIGPO/(PI*R*R)
  601 H60=A1*A4*(B8+B7*PA)+E60*PO
      H42=D42+E42*PO+B7*Q2*PA
      H24=D24+E24*PO+A2*A3*B7*PA
      H06=A2*A3*B6+E06*PO
      H40=D40+A1*A7*PO*(B8+B7*(PO+PA))
      HC40=Q2/R-R*A1*A7*PO
      H22=D22+PO*(E22+EB22*PO+A3*A7*B7*PA)
      HC22=A3*(A7*R*PO+6.*A2/R)
      H04=D04+PO*(E04+EB04*PO)
      HS21=F2*A3*(A7*PO-F2*A2/R**F2)
      HC20=A3*(PO*A7/R-A2/R**F3)
      H02=A3*A7*B4*PO
C     STABILITY DETERMINANT CONSTANTS (EQUA.45)
       DO 227 N=1,6
      B1(N)=-HC60*V(6)*U(3)-HC42*V(4)*U(3)*S(N,2)-HC24*V(2)*U(3)*S(N,4)
      B1(N)=B1(N)+HC40*V(4)*U(1)+HC22*V(2)*U(1)*S(N,2)-HC20*V(2)*R
      B2(N)=HS21*V(2)*S(N,1)-HS41*V(4)*U(2)*S(N,1)-HS23*V(2)*U(2)*S(N,3)
      B3(N)=H80*V(8)*U(5)+H62*V(6)*U(5)*S(N,2)+H44*V(4)*U(5)*S(N,4)
      B3(N)=B3(N)+H26*V(2)*U(5)*S(N,6)+H08*U(5)*S(N,8)-H60*V(6)*U(3)
      B3(N)=B3(N)+H42*V(4)*U(3)*S(N,2)-H24*V(2)*U(3)*S(N,4)
      B3(N)=B3(N)-H06*U(3)*S(N,6)+H40*V(4)/R+H22*V(2)*S(N,2)/R
      B3(N)=B3(N)+H04*S(N,4)/R-H02*R*S(N,2)-H20*V(2)*R
C     CONSTANTS FOR CHARACTERISTIC EQUATION (EQUA. 49)
  227 X(N)=F2*B3(N)
      DO 228 N=1,5
  228 Y(N)=(B1(N)+B1(N+1)-B2(N)+B2(N+1))**2
      TO=X(1)*X(2)*X(3)*X(4)*X(5)*X(6)
      T1=-X(1)*X(2)*(X(3)*X(4)*Y(5)+X(3)*X(6)*Y(4)+X(5)*X(6)*Y(3))
      T1=T1-X(4)*X(5)*X(6)*(X(1)*Y(2)+X(3)*Y(1))
      T2=X(1)*(Y(5)*(X(2)*Y(3)+X(4)*Y(2))+X(6)*Y(2)*Y(4))
      T2=T2+Y(1)*(X(3)*(X(4)*Y(5)+X(6)*Y(4))+X(5)*X(6)*Y(3))
      T3=-Y(1)*Y(3)*Y(5)
C     SOLUTION OF CHARACTERISTIC EQUATION (EQUA. 48)
      Q=(F3*T1/T3-(T2/T3)**2 )/F3
      T=(F2*(T2/T3)**3 -9.*T1*T2/T3**2 +27.*TO/T3)/27.
      Z=T**2 /F4+Q**3 /27.
      IF(Z)250,260,270
  270 BIGA=(Z**.5-T/F2)**(F1/F3)
      BIGB=(-(Z**.5)-T/F2)**(F1/F3)
      EMA2=BIGA+BIGB
      IF(EMA2)390,400,400
  260 BIGA=(-T/F2)**(F1/F3)
      EMA21=F2*BIGA
      EMA22=-BIGA
      IF(EMA21)261,262,262
  261 EMA2=EMA22
      GO TO 400
```

```
  262 EMA2=EMA21
      GO TO 400
  250 THETA=ATANF((-(Q**3 /27.)-(T**2 /F4))**.5/(-T/F2))
      Q3=F2*(-Q/F3)**.5
      EMA21=Q3*COSF(THETA/F3)
      EMA22=Q3*COSF(THETA/F3+F2*PI/F3)
      EMA23=Q3*COSF(THETA/F3+F4*PI/F3)
      IF(EMA21)251,252,252
  251 IF(EMA22)258,259,259
  258 IF(EMA23)390,282,282
  282 EMA2=EMA23
      GO TO 400
  259 IF(EMA22-EMA23)280,280,284
  284 IF(EMA23)280,281,281
  280 EMA2=EMA22
      GO TO 400
  281 EMA2=EMA23
      GO TO 400
  252 IF(EMA21-EMA22)253,253,285
  285 IF(EMA22)253,254,254
  253 EEMA2=EMA21
      GO TO 255
  254 EEMA2=EMA22
  255 IF(EEMA2-EMA23)256,256,286
  286 IF(EMA23)256,257,257
  257 EMA2=EMA23
      GO TO 400
  256 EMA2=EEMA2
  400 EMA=EMA2**.5
      EMO=EMA*PI*R**F2
      PUNCH513,BIGPO,EMO
      GO TO 391
  390 PUNCH516,BIGPO
C     BEGIN CYCLING OF INPUT DATA
  391 PA=PA+PAINC
      BIGPO=BIGPO+BGPOI
      IF(PA-PAMAX)601,601,201
  201 PO=PO+POINC
      IF(PO-POMAX)600,600,202
  202 A1A=A1A+A1AIN
      IF(A1A-A1AMX)60,60,203
  203 A2A=A2A+A2AIN
      IF(A2A-A2AMX)55,55,204
  204 D1D=D1D+D1DIN
      IF(D1D-D1DMX)50,50,205
  205 D2D=D2D+D2DIN
      IF(D2D-D2DMX)45,45,206
  206 ROH=ROH+ROHIN
      IF(ROH-ROHMX)35,35,207
  207 RPL=RPL+RPLIN
      IF(RPL-RPLMX)25,25,208
  208 EM=EM+EMINC
      IF(EM-EMMAX)15,15,209
  209 EK=EK+EKINC
```

```
      IF(EK-EKMAX)10,10,210
C     OUTPUT STATEMENT
  210 PRINT 101
  101 FORMAT(13HLOAD NEW DATA)
  501 FORMAT(8F8.0)
  510 FORMAT(20HE,VXS,VSX,RAD,K,M = ,E8.2,2F5.2,F8.2,2F5.1)
  511 FORMAT(6X22HA1/A,A2/A,D1/D,D2/D = ,F6.2,3F7.2)
  512 FORMAT(2X20HPI X RAD / LENGTH = ,F6.3)
  513 FORMAT(10X11HBIGPO,MO = ,E9.2,3XE13.6)
  514 FORMAT(8X11HPRESSURE = ,F8.2)
  515 FORMAT(4X18HRAD / THICKNESS = ,F9.2)
  516 FORMAT(10X11HBIGPO,MO = ,E9.2,6X4HIMAG)
      GO TO 100
      END
```

APPENDIX E

COMPUTER PROGRAM OUTPUT FORMAT

INPUT DATA FOR THE FOLLOWING OUT-PUT FORMAT

```
30000000.3     .3      10.

15.      15.      30.      0.      5000.    10000.

1200.    6.       1.       1.      1.       1.       1.       1.

400.     2.       1.       1.      1.       1.       1.       1.

1600.    8.       1.       1.      1.       1.       1.       1.
```

OUT-PUT FORMAT

```
E,VXS,VSX,RAD,K,M =   .30E+08   .30   .30    10.00   1.0   1.0
   PI X RAD / LENGTH =   6.000
     RAD / THICKNESS =    1200.00
       A1/A,A2/A,D1/D,D2/D =    1.00    1.00    1.00    1.00
         PRESSURE =     15.00
           BIGPO,MO =    .00E-99     4.145080E+08
           BIGPO,MO =   5.00E+03     4.145072E+08
           BIGPO,MO =   1.00E+04     4.145060E+08
         PRESSURE =     30.00
           BIGPO,MO =    .00E-99     4.145147E+08
           BIGPO,MO =   5.00E+03     4.145134E+08
           BIGPO,MO =   1.00E+04     4.145121E+08
     RAD / THICKNESS =    1600.00
       A1/A,A2/A,D1/D,D2/D =    1.00    1.00    1.00    1.00
         PRESSURE =     15.00
           BIGPO,MO =    .00E-99     3.108863E+08
           BIGPO,MO =   5.00E+03     3.108854E+08
           BIGPO,MO =   1.00E+04     3.108838E+08
         PRESSURE =     30.00
           BIGPO,MO =    .00E-99     3.108928E+08
           BIGPO,MO =   5.00E+03     3.108916E+08
           BIGPO,MO =   1.00E+04     3.108903E+08
   PI X RAD / LENGTH =   8.000
     RAD / THICKNESS =    1200.00
       A1/A,A2/A,D1/D,D2/D =    1.00    1.00    1.00    1.00
         PRESSURE =     15.00
           BIGPO,MO =    .00E-99     2.344654E+08
           BIGPO,MO =   5.00E+03     2.344708E+08
           BIGPO,MO =   1.00E+04     2.344760E+08
         PRESSURE =     30.00
           BIGPO,MO =    .00E-99     2.344727E+08
           BIGPO,MO =   5.00E+03     2.344779E+08
           BIGPO,MO =   1.00E+04     2.344833E+08
```

```
RAD / THICKNESS =    1600.00
  A1/A,A2/A,D1/D,D2/D =    1.00    1.00    1.00    1.00
    PRESSURE =     15.00
      BIGPO,MO =    .00E-99     1.758522E+08
      BIGPO,MO =   5.00E+03     1.758575E+08
      BIGPO,MO =   1.00E+04     1.758626E+08
    PRESSURE =     30.00
      BIGPO,MO =    .00E-99     1.758593E+08
      BIGPO,MO =   5.00E+03     1.758646E+08
      BIGPO,MO =   1.00E+04     1.758698E+08
```

APPENDIX F
PARTIAL LIST OF DEFINITIONS OF COMPUTER PROGRAM SYMBOLS

E	Modulus of elasticity for isotropic case.
VXS	Poissoin's ratio for the x to s direction.
VSX	Poisson's ratio for the s to x direction.
R	Radius of the shell.
A1A	$= \alpha_1/\alpha$; where $\alpha = Eh/2\,(1 - \mu_{xs}\mu_{sx})$
A2A	$= \alpha_2/\alpha$
D1D	$= D_1/D$; where $D = Eh^3/24\,(1 - \mu_{xs}\mu_{sx})$
D2D	$= D_2/D$
ROH	$= R/h$
RPL	$= \pi R/L$
EK	$= k$
EM	$= m$
PO	$= p_o$ radial pressure
BIGPO	$= P_o$ axial load
PA	$= P_a = P_o/\pi R^2$
EMO	$= M_o$ end moment
EMA	$= M_o/\pi R^2$
AA1A	- initial value of A1A (minimum)
AA2A	- initial value of A2A (minimum)
AD1D	- initial value of D1D (minimum)
AD2D	- initial value of D2D (minimum)
AROH	- initial value of ROH (minimum)
ARPL	- initial value of RPL (minimum)
AEK	- initial value of EK (minimum)

AEM	-	initial value of EM (minimum)
APO	-	initial value of PO (minimum)
ABGPO	-	initial value of BIGPO (minimum)
A1AMX	-	final value of A1A (maximum)
A2AMX	-	final value of A2A (maximum)
D1DMX	-	final value of D1D (maximum)
D2DMX	-	final value of D2D (maximum)
ROHMX	-	final value of ROH (maximum)
RPLMX	-	final value of RPL (maximum)
EKMAX	-	final value of EK (maximum)
EMMAX	-	final value of EM (maximum)
POMAX	-	final value of PO (maximum)
BGPOM	-	final value of BIGPO (maximum)
A1AIN	-	incrument of A1A
A2AIN	-	incrument of A2A
D1DIN	-	incrument of D1D
D2DIN	-	incrument of D2D
ROHIN	-	incrument of ROH
RPLIN	-	incrument of RPL
EKINC	-	incrument of EK
EMINC	-	incrument of EM
POINC	-	incrument of PO
BGPOI	-	incrument of BIGPO
EL		Length of shell
H		Thickness of the shell